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Distributing points on the sphere:

Partitions, separation, quadrature and energy

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Globe, Bronwyn Oliver, 2002 (Courtesy of the artist and Roslyn Oxley9 Gallery).

DISTRIBUTING POINTS ON THE SPHERE

A thesis submitted for the degree of

Doctor of Philosophy

By

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ORIGINALITY STATEMENT

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I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project’s design and conception or in style, presentation and linguistic expression is acknowledged.’

Paul Leopardi

For Margaret Rose

Abstract

This thesis concentrates on a set of problems and approaches relating to the generation and analysis of spherical codes.

The work was conducted at the University of New South Wales between 2002 and 2006, during a short visit to Vanderbilt University in 2004, and at the University of Sydney in 2006.

The key results include:

1. A description of an equal area partition of the unit sphere $\mathbb{S}^d$ called the EQ partition.
2. A detailed description of the EQ algorithm which produces the EQ partition.
3. Proofs that EQ partitions are equal area partitions with small diameter.
4. A detailed description of the construction of a spherical code called the EQ code, based on the EQ partition.
5. A proof that the sequence of EQ codes is well separated.
6. An examination of the suitability of the EQ codes for polynomial interpolation.
7. An examination of the packing density of the EQ codes.
8. Modified constructions of the EQ codes to allow nesting or to maximize the packing radius.
9. A scheme to use the EQ partitions and EQ codes for spherical coding and decoding.

10. A proof that for $0 < s < d$ a sequence of $\mathbb{S}^d$ codes which is well separated and weak star convergent has a Riesz s energy which converges to the corresponding energy double integral.
11. A bound on the rate of convergence of Riesz s energy given the rate of convergence to zero of the spherical cap discrepancy.
12. A comparison of Coulomb energy estimate for $\mathbb{S}^2$ spherical designs given in [73] with estimates obtained using only the separation and the estimated spherical cap discrepancy of the spherical designs.
13. A proof that the EQ codes are weak star convergent.
14. Estimates of the rate of convergence to zero of the spherical cap discrepancy of EQ codes.
15. Estimates of the rate of convergence of Riesz s energy of EQ codes to the energy double integral.

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I would like to thank my collaborators, Kirsten Hesse for the work on energy and spherical designs, and Walter Gautschi for joint work on Jacobi polynomials which will be published elsewhere.

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Ed Saff and Ian Sloan posed the problem relating to the equal area partition of the unit sphere $\mathbb{S}^3$, which was the basis of Chapter 3. The partition algorithm featured in Chapter 3 draws heavily on the algorithm described in the PhD thesis of Yanmu Zhou [167]. A description of the algorithm appears in [99]. The reviewers of [99] helped to make that paper and this thesis much better.

The proofs in Chapter 5 relating to weak* convergence and separation benefited greatly from discussions with Ed Saff and Doug Hardin.

The symbolic calculations which support this thesis benefited greatly from the use of Maple [104]. The EQ Sphere Partitioning Toolbox highlighted in Chapter 3 would not have been possible without Matlab [154].

My examiners have suggested a number of corrections and improvements to this thesis. In most cases I have taken their advice. Any remaining errors are my own.

Finally, I would like to thank my sister Christine and my late parents, Romeo and Margaret for the love and support they have given me through my life and through this, my latest of many periods of academic study.

Contents

List of Tables

List of Figures

List of Symbols

$\boxdot$	Centre point of a region
$\obar$	Meridian of longitude
$\odot$	Map from spherical to Cartesian coordinates
$\ominus$	Parallel of colatitude
$\oslash$	Maximum diameter bound
$\to$	Maps to, converges to
a	Rounding term in number of regions in collar
$\mathcal{A}$	Maximum number of codepoints
β	Analogue of rounding term at bottom of collar
b	Bottom colatitude
$\mathbb{B}$	Unit ball
$\mathcal{B}$	Analogue of bottom colatitude
C	Constant
$\mathbb{C}$	Complex number field
γ	1 plus a small factor used in area calculations
Γ	Gamma function
$\mathbb{D}$	Feasible domain
$\mathcal{D}$	Dimension of spherical polynomial space
disc	Spherical cap discrepancy
d	Dimension of sphere
db	Per-region diameter bound
diam	Diameter
Δ	Lower bound on minimum distance between codepoints

δ	Angle between parallels defining a collar
Δ	Analogue of angle between parallels defining a collar
∂	Boundary
$\mathcal{E}$	Minimum energy
EQP	Recursive zonal equal area spherical code
EQ	Recursive zonal equal area partition
E	Energy functional
$\mathcal{F}$	Facet
g	Counting function
$\mathcal{H}$	Class of half spaces
η	A small factor used in area calculations
Θ	Spherical radius of cap of a given area
θ	Spherical radius of cap
ϑ	Colatitude
$\mathcal{I}$	Normalized spherical double integral
$\mathcal{J}$	Normalized spherical single integral
J	Function used in estimates of colatitudes
j	Order of spherical harmonic
$\mathcal{L}$	Minimum index
L^∞	Space of bounded real functions
Λ	Polynomial interpolation operator
m	Number of regions in collar
$\mathcal{M}$	Analogue of number of regions in collar
maxdiam	Maximum diameter
mod	Modulo
μ	Polynomial degree
n	Number of collars
$\mathbb{N}$	Natural numbers starting from 1
$\mathcal{N}_0$	Analogue of the number of points for a given ideal number of collars

$\mathcal{N}$	Number of codepoints
ν	Analogue of ideal number of collars
$\mathbf{o}$	North Pole
O	Order
p	Scaled diameter bound
$\mathbb{P}$	Polynomial space
$\mathcal{P}$	Analogue of scaled diameter bound
Π	Equatorial map
ϖ	Stereographic projection
R	Region
$\mathbb{R}$	Real number field
$\mathcal{R}$	Map a pair of spherical polar coordinates to a region
round	Round to nearest integer
$\overset{*}{\sigma}$	Normalized Lebesgue area measure
S	Spherical cap
s	Spherical distance
$\mathbb{S}$	Sphere
sinc	Sinc function
σ	Lebesgue area measure on the unit sphere
$\mathcal{T}$	Analogue of colatitude of top of collar
τ	Analogue of rounding term at top of collar
T	Maximum Euclidean distance
t	Strength of quadrature rule
U	Riesz potential
u	Modified cutoff Riesz potential
U	Cutoff Riesz potential
u	Generic potential
Υ^{-1}	Map from Euclidean to spherical distance
Υ	Map from spherical to Euclidean distance

$\overset{*}{\mathcal{V}}$	Normalized area of a spherical cap
$\mathcal{V}$	Area of a spherical cap
Φ	The reproducing kernel of a space of spherical polynomials
ϕ	Generic kernel
w	Maximum Euclidean radius of collar
$\mathcal{W}$	Analogue of maximum Euclidean radius of collar
y	Ideal number of regions in collar
$\mathcal{Y}$	Analogue of ideal number of regions in collar
Y	Spherical harmonic polynomial
$\mathbb{Z}$	Integers
Ψ	Analogue of factor used to calculate minimum distance to side facet
ψ	Factor used to calculate minimum distance to side facet
ω_d	Area of $\mathbb{S}^d$
ω_{d-1}	Area of $\mathbb{S}^{d-1}$

Chapter 1

Introduction

"Il meglio è l'inimico del bene."

(The best is the enemy of the good.)

– Voltaire, [158, "Art Dramatique"], translated in [157].

1.1 Good distributions of points on the sphere

This thesis explores the connections between a number of topics related to the distribution of points on the unit sphere

$$\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} \quad | \; \|\mathbf{x}\| = 1\}, \tag{1.1.1}$$

where $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$, ie.

$$\|\mathbf{x}\|^2 = \sum_{k=1}^{d+1} x_k^2. \tag{1.1.2}$$

The unit sphere $\mathbb{S}^d$ defined by (1.1.1) is embedded in the vector space $\mathbb{R}^{d+1}$ with centre at the origin.

The topics which are explored in this thesis include *partitions*, *approximation*, *interpolation*, *quadrature* and *energy*.

The distribution of points on the unit sphere is a subject area which has many applications and which gives rise to a number of problems, many of which are unsolved or are *hard* in the sense of computational complexity. Two major fields of study which involve problems of point distributions on the unit sphere are approximation theory and coding theory. These are discussed further in the following sections. The study of the distribution of points on the unit sphere also arises naturally in the fields of quantum information theory [125, 127, 126] and number theory.

One key question addressed in this thesis is

"What is meant by a *good distribution* of points on the unit sphere?"

It has long been known that for $d > 1$ there is usually no single answer to this question, in the sense that the most appropriate definition of goodness depends on the requirements of the problem to be solved. For example, a set of 169 points which approximately minimizes the Coulomb energy on $\mathbb{S}^2$ performs very poorly when used to interpolate spherical polynomials of total degree at most 12. See [162, pp. 212–217] for properties of the set computed by Fliege and Maier [56, p. 26] and [161] for the set of lower energy computed by Womersley.

Here, as usual in this thesis, we treat any two finite subsets $X, Y \subset \mathbb{S}^d$ as equivalent if X can be mapped onto Y by an orthogonal transformation in $\mathbb{R}^{d+1}$.

In many cases it is easier to reason about *sequences* of finite subsets of the unit sphere, rather than a single finite subset in isolation. For example, we can use various definitions of *discrepancy* [6, 64, 38] [30, Chapter 2] and convergence to define the concept of *asymptotic equidistribution* of sequences of finite subsets of the unit sphere [168, 39].

Also, it is often sufficient to find a sequence of finite subsets which is merely *close enough* to optimal with respect to some criterion of goodness, rather than optimal. This is because the associated optimization problem may be hard in the sense of computational complexity [76, p. 333], and often exhibits many local minima or maxima [134, p. 7]. Thus in many cases it is useful to find a construction for finite subsets of the unit sphere which is reasonably fast and which is reasonably good with respect to one criterion or more.

1.2 Approximation on the sphere

The generation and evaluation of good finite subsets of the unit sphere is related in a number of ways to the subject of approximation on the sphere.

The theory and practice of approximation on the unit sphere is a vast field with many areas of application [70, 130]. These areas include computational chemistry [163] , physics [106], astronomy [152, 111, 153], planetary science [97], geosciences [51, Chapters 2, 4, 7 and 8] [58, 13, 59], medical imaging [124, Chapter 8], computer vision [60], sensory physiology [80], bioinformatics [109] and biology [29].

The generic problems of approximation on the unit sphere include:

- Given a finite subset of the unit sphere, with corresponding function values, find the closest member of a defined function set eg. [107];
- Given a function defined at all points on the unit sphere, or a defined subset of the unit sphere, find the closest member of a defined function set;
- Given a differential or integral equation on the unit sphere, or a defined subset of the unit sphere, find an approximate solution, eg. [110].

For some related surveys, see [52, 13].

1.3 Sphere packing, coding theory and communications theory

The connection between communications theory and the study of finite subsets of higher dimensional unit spheres arguably began with Shannon's seminal paper of 1959 [140]. This connection is so strong that a finite subset of the unit sphere is often called a *spherical code*. This thesis adopts this terminology.

The study of spherical codes often emphasizes their *packing* and *covering* properties. Packing can be defined in terms of the minimum distance between the points of a spherical code, and leads to the concept of a *well separated* sequence of codes. Covering can be defined in terms of the maximum of the distance between any point on the unit sphere and a point of the code, also known as the *mesh norm*.

The study of packings and coverings on the unit sphere well predates Shannon and could be said to have begun with the kissing problem of Newton and Gregory in 1694 [149, Chapter 5] [119, p. 875].

For a survey on coding theory which mentions its connections to spherical codes and quantum information theory, see [26]. Other relevant papers on communications theory, coding theory and spherical codes include [164, 165, 66, 67, 82, 68, 156].

1.4 Energy as a criterion of goodness

Given a potential, usually a decreasing function of distance, we can define the energy of a spherical code. For example, the Riesz s energy is defined using the Riesz r^{-s} potential [39, 24].

Even though a spherical code has minimal energy for a given potential, that does not mean that such a code is universally "good". For example, for any fixed $d > 1$ there are very few $\mathbb{S}^d$ codes which are known to minimize all *completely monotonic* potentials. The number of such codes currently known is literally just a handful: between 2 and 5 [33, Table 1, Theorem 1.2].

It is well known that for $0 < s < d$ an increasing sequence of minimal s energy finite subsets of $\mathbb{S}^d$ is asymptotically equidistributed [39]. For $d - 1 < s < d$ such a sequence is also well separated [89]. In this thesis, we examine the concepts of equidistribution, separation and energy from a different direction. We show that for $0 < s < d$, if a sequence of $\mathbb{S}^d$ codes is well separated and asymptotically equidistributed, it has a well behaved s energy. A major part of this thesis is devoted to the construction of just such a well-behaved sequence and the exploration of some of its properties.

1.5 Organization and key results of this thesis

This thesis concentrates on a subset of problems and approaches relating to the generation and analysis of spherical codes. The focus is on theory, construction and computation rather than on particular applications. The organization and key results of this thesis are listed below.

Chapter 2 of this thesis gives an overview and detailed definitions of the concepts and structures which are addressed in this thesis.

Chapter 3 of this thesis describes an *equal area partition* of $\mathbb{S}^d$. Building on a partition algorithm for the unit sphere $\mathbb{S}^2$, as described by Rakh-manov, Saff and Zhou [120] and Zhou [167], and an outline of a construction for $\mathbb{S}^3$, as discussed by Saff [133] and Sloan [141] during 2003 to 2005, the *EQ* algorithm partitions a unit sphere in any dimension into regions of equal area and small diameter. The partition is called an EQ partition. The Matlab implementation of the EQ algorithm is available via SourceForge [98]. Chapter 3 contains a detailed description of the EQ algorithm and contains proofs that the EQ partitions are equal area partitions with small diameter.

Chapter 4 of this thesis describes a spherical code, the *EQ code*, which uses the EQ partition for its construction. Chapter 4 contains a detailed description of the construction of the EQ code and a proof that the sequence of EQ codes is well separated. Chapter 4 also

- examines the suitability of the EQ codes for polynomial interpolation,
- examines the packing density of the EQ codes,
- examines modified constructions to allow nesting or to maximize the packing radius, and
- examines a scheme to use the EQ partitions and EQ codes for spherical coding and decoding.

Chapter 5 of this thesis explores the relationships between *weak star convergence*, spherical cap discrepancy, minimum separation and energy of spherical codes. A sequence of spherical codes converges in the weak star sense if the corresponding equal weighted quadrature rules converge to the integral for all continuous functions on the sphere. The normalized spherical cap discrepancy of a spherical code is the supremum over all spherical caps of the difference between the normalized area of the cap and the proportion of points of the code which lie in the cap. Weak star convergence is equivalent to convergence to zero of normalized spherical cap discrepancy. Chapter 5 contains

- a proof that for $0 < s < d$ a sequence of $\mathbb{S}^d$ codes which is well separated and weak star convergent has a Riesz s energy which converges to the corresponding energy double integral,
- a bound on the rate of convergence of energy given the rate of convergence to zero of the normalized spherical cap discrepancy,
- a proof that the EQ codes are weak star convergent, estimates of the rate of convergence to zero of the normalized spherical cap discrepancy, and estimates of the rate of convergence of Riesz s energy to the energy double integral.

Recall that a spherical t-design is an equal weighted quadrature rule on the unit sphere which is exact for all polynomials of degree up to t. In joint work with Hesse [73] the author proved that for a well separated sequence of spherical designs on $\mathbb{S}^2$ such that each t-design has $(t+1)^2$ points, the Coulomb energy has the same first term and a second term of the same order as the minimum Coulomb energy for $\mathbb{S}^2$ codes. Chapter 5 contains a comparison of this earlier energy estimate with estimates obtained using only the separation and the estimated normalized spherical cap discrepancy of these spherical designs.

Chapter 2

Preliminaries

"... plano vero sectum sphaericum circulum sectione repraesentat, mentis creatae, quae corpori regendo sit praefecta, genuinam imaginem, quae in ea proportione sit ad sphaericum, ut est mens humana ad divanam, ..."
(... when intersected by a plane, the sphere displays in this section the circle, the genuine image of the created mind, placed in command of the body which it is appointed to rule; and this circle is to the sphere as the human mind is to the Mind Divine, ...)

– Kepler, [83, Book IV, pp. 119-120], quoted and translated in Pauli [117, p. 161].

2.1 Some notation

This section describes some of the notation used in this thesis.

Sequences of spherical codes.

This thesis considers sequences of $\mathbb{S}^d$ codes of the form $\mathcal{X} = (X_1, X_2, \ldots)$, with each $\mathbb{S}^d$ code X_ℓ being a finite subset of $\mathbb{S}^d$ of the form $X_\ell := \{\mathbf{x}_{\ell,1}, \ldots, \mathbf{x}_{\ell,\mathcal{N}_\ell}\} \subset \mathbb{S}^d$, with the points of X_ℓ distinct, so that $\mathcal{N}_\ell := |X_\ell|$.

The points of a spherical code are usually called *codepoints* as a reminder that the code is a finite set and to distinguish between the codepoints and other points or subsets of the unit sphere.

Intervals.

This thesis uses the standard notation for intervals on the real line, augmented by a small amount of arithmetic for the purpose of abbreviation.

For example, the statement $x \in [a, b)\ c$ (where $c > 0$) means $a\,c \leqslant x < b\,c$.

Monotonicity and limits.

The notation $f(x) \nearrow y$ as $x \to \infty$ means that $f(x)$ is asymptotically monotonically increasing with x, with limit y.

Similarly, $f(x) \searrow y$ as $x \to \infty$ means that $f(x)$ is asymptotically monotonically decreasing with x, with limit y.

2.2 Trigonometric functions and the Gamma function

Before plunging into the geometry of the sphere and related topics, we list some properties, identities and estimates related to the well-known trigonometric functions and the Gamma function [2, Chapter 6] [4, Chapter 1]. These prove to be useful in estimates relating to various aspects of geometry on the sphere.

Trigonometric functions.

The following identities, the *addition formulae* for trigonometric functions are well known and are used throughout this thesis.

For all real θ, ϕ, the identities

$$\sin(\theta + \phi) = \sin\theta\ \cos\phi + \cos\theta\ \sin\phi, \quad \cos(\theta + \phi) = \cos\theta\ \cos\phi - \sin\theta\ \sin\phi \tag{2.2.1}$$

hold. These identities are easily proven using complex multiplication and *Euler's formula* which states that $e^{i\theta} = \cos\theta + i\ \sin\theta$ for all real θ.

Later we will need to compare $\sin\theta$ with $\sin(\theta+\phi)$, for various θ and ϕ. The following estimate is useful for this task.

Lemma 2.2.1. *For all $\theta, \phi \in \mathbb{R}$ we have*

$$\sin(\theta+\phi) - \sin\theta = 2\ \sin\frac{\phi}{2}\ \cos\left(\theta + \frac{\phi}{2}\right). \tag{2.2.2}$$

Therefore for $\phi \in (0,\pi]$, $\theta \in (0, \pi/2 - \phi/2]$ we have $\sin(\theta+\phi) > \sin\theta > 0$.

We will also need the well known estimates

$$|\cos\theta| \leqslant 1, \tag{2.2.3}$$

and for $\ell > 0$, $\theta \neq 0$,

$$\sum_{k=0}^{2\ell-1} (-1)^k \frac{\theta^{2k}}{(2k)!} < \cos\theta < \sum_{k=0}^{2\ell} (-1)^k \frac{\theta^{2k}}{(2k)!}. \tag{2.2.4}$$

The inequality (2.2.4) can be proven using Taylor's theorem with the Lagrange formula for the explicit remainder term [135, Theorem 11.6.1 and Corr. 11.6.2, pp. 730-731]. A simple proof of the non-strict version of this inequality appears in [14].

In the estimate below we assume that $\theta \in (0,\xi]$, $\xi \in (0,\pi/2]$, and use the well-known function

$$\operatorname{sinc}\theta := \frac{\sin\theta}{\theta}. \tag{2.2.5}$$

We have the well-known estimate

$$\sin\theta \in [\operatorname{sinc}\xi, 1]\ \theta. \tag{2.2.6}$$

The Gamma function.

Lemma 2.2.2. *The Gamma function has the following well-known properties. For proofs see the references given with each property.*

1. *For $x > 0$,*

$$\Gamma\,(x+1) = x\,\Gamma(x). \tag{2.2.7}$$

For proof, see [4, (1.1.6)].

2. *See [146, Chapter 13, Problem 1 (b), p. 287] or [4, (1.1.7), (1.1.8)] for proof that*

$$\Gamma(1) = \Gamma(2) = 1. \tag{2.2.8}$$

3. *See [146, Chapter 13, Problem 4, p. 288] or [4, (1.1.22)] for proof that*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.2.9}$$

4. *For $x > 0$, $\log\Gamma(x)$ is a strictly convex function of x. That is, for $x, y > 0$, $a \in (0,1)$,*

$$\log\Gamma\,\big(ax + (1-a)y\big) < a\,\log\Gamma(x) + (1-a)\,\log\Gamma(y). \tag{2.2.10}$$

This is called the log-convexity *of the Gamma function. For proof, see [132, Chapter 6, Theorem 8.18, p. 192] or [4, Corollary 1.2.6, p. 13].*

5. *For $x > 0$ we have*

$$\Gamma(2x)\ \Gamma\left(\frac{1}{2}\right) = 2^{2x-1}\ \Gamma(x)\ \Gamma\left(x + \frac{1}{2}\right). \tag{2.2.11}$$

This is called the Legendre duplication formula. *For proof, see [4, Theorem 1.5.1, p. 22].*

We now have the following estimates.

Lemma 2.2.3. *For $x \geqslant \frac{3}{2}$ we have*

$$\frac{d}{dx}\log\Gamma(x) > 0. \tag{2.2.12}$$

Lemma 2.2.4. *For $x \geqslant 1$ we have*

$$\Gamma(2x) \geqslant 4^{x-1}\,\Gamma(x), \tag{2.2.13}$$

with equality only when $x = 1$.

Lemma 2.2.5. *For $x > 0$ we have*

$$\frac{\Gamma\left(x + \frac{1}{2}\right)}{\Gamma(x)} < \sqrt{x}. \tag{2.2.14}$$

Lemma 2.2.6. *For $x \geqslant 1$ we have*

$$\Gamma(x+1) \leqslant x^x. \tag{2.2.15}$$

with equality only when $x = 1$.

2.3 The geometry of the unit sphere

This section describes some well known but essential aspects of the geometry of $\mathbb{S}^d$.

2.3.1 Small and great spheres and circles, generalized spheres

For the sphere $\mathbb{S}^d$, for $k \in \{2, \ldots, d\}$, a *small sphere* or small $\mathbb{S}^k$ is a non-empty intersection of $\mathbb{S}^d$ with a k dimensional hyperplane, and a *great k-sphere* or great $\mathbb{S}^k$ is the intersection of $\mathbb{S}^d$ with a k dimensional hyperplane through the origin of $\mathbb{R}^{d+1}$. If the hyperplane is 2 dimensional, we have a *small circle* or a *great circle*, respectively. If the hyperplane is d dimensional, we have a *small sphere* or a *great sphere*, respectively.

When this thesis uses the term "hyperplane" without qualification, this generally means a d dimensional hyperplane in $\mathbb{R}^{d+1}$.

A *generalized sphere* is either a sphere or a hyperplane.

2.3.2 Euclidean and spherical distances

The Euclidean distance between two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ is defined via the $\mathbb{R}^{d+1}$ norm to be $\|\mathbf{a} - \mathbf{b}\|$.

The Euclidean inner product of two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ is the usual inner product in $\mathbb{R}^{d+1}$, namely

$$\mathbf{a} \cdot \mathbf{b} := \sum_{k=1}^{d+1} a_k b_k. \tag{2.3.1}$$

We have $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} = 1$ for any $\mathbf{x} \in \mathbb{S}^d$.

Definition 2.3.1. *The spherical distance* $\mathsf{s}(\mathbf{a}, \mathbf{b})$ *of two points* $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ *is defined as*

$$\mathsf{s}(\mathbf{a}, \mathbf{b}) := \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

where the inner product is that of (2.3.1).

We extend this definition to distances between a point and a set, and distances between sets. For example, $\mathsf{s}(\mathbf{x}, Y)$ is the infimum of the spherical distance between point $\mathbf{x}$ and any point of the set $Y \subset \mathbb{S}^d$.

We now recall a number of well known results relating to spherical distance. First, for $\theta \in [0, \pi]$ we define the function

$$\Upsilon(\theta) := 2 \sin \frac{\theta}{2}. \tag{2.3.2}$$

Lemma 2.3.2. *For* $\mathbb{S}^d$*,*

1. *The geodesics are great circles. More precisely, every geodesic is a constant speed parameterization of an arc of a great circle.*
2. *The curve of shortest arc length between two points is an arc of a great circle, with arc length up to* π*.*
3. *Spherical distance is the same as geodesic arc length, up to* π*.*
4. *The relationship between Euclidean distance and spherical distance is independent of position in the following sense. For any two points* $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$*,*

$$\|\mathbf{a} - \mathbf{b}\| = \Upsilon\left(\mathsf{s}(\mathbf{a}, \mathbf{b})\right), \tag{2.3.3}$$

where the function Υ *is defined by (2.3.2).*

5. *The function* Υ *defined by (2.3.2) is continuous on* $[0, \pi]$ *and monotonic increasing on* $[0, \pi)$.

6. *For* $\alpha, \beta, \alpha + \beta \in [0, \pi]$ *we have*

$$\Upsilon(\alpha + \beta) \leqslant \Upsilon(\alpha) + \Upsilon(\beta) \tag{2.3.4}$$

with equality only when $\alpha = 0$ *or* $\beta = 0$.

7. *As a result of Property 5, the function* Υ *has an inverse which is defined on* $[0, 2]$. *For this inverse function* Υ^{-1}, *for* $a, b, a + b \in [0, 2]$ *we have*

$$\Upsilon^{-1}(a) + \Upsilon^{-1}(b) \leqslant \Upsilon^{-1}(a + b), \tag{2.3.5}$$

with equality only when $a = 0$ *or* $b = 0$.

8. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$, $\|\mathbf{a} - \mathbf{b}\| \leqslant \mathsf{s}(\mathbf{a}, \mathbf{b})$, *with equality only when* $\mathbf{a} = \mathbf{b}$.

9. *For* $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$,

$$\lim_{\mathbf{a} \to \mathbf{b}} \frac{\|\mathbf{a} - \mathbf{b}\|}{\mathsf{s}(\mathbf{a}, \mathbf{b})} = 1. \tag{2.3.6}$$

2.3.3 Spherical polar coordinates

Spherical polar coordinates describe a point $\mathbf{a}$ on $\mathbb{S}^d$ by using one longitude, $\alpha_1 \in \mathbb{R}$, and $d-1$ colatitudes, $\alpha_i \in [0, \pi]$, for $i \in \{2, \dots, d\}$. The longitude α_1 is taken modulo 2π so that eg. the coordinates $(0, \alpha_2, \dots, \alpha_d)$ and $(2\pi, \alpha_2, \dots, \alpha_d)$ describe the same point. The unit sphere $\mathbb{S}^d$ defined by (1.1.1) is embedded in the vector space $\mathbb{R}^{d+1}$ with centre at the origin.

A point $\mathbf{a} \in \mathbb{S}^d$ can therefore be described by its spherical polar coordinates or by its corresponding Cartesian coordinate vector.

Definition 2.3.3. *We define the spherical polar to Cartesian coordinate map* $\odot$ *by*

$$\odot : \mathbb{R} \times [0, \pi]^{d-1} \to \mathbb{S}^d \subset \mathbb{R}^{d+1},$$

$$\odot(\alpha_1, \alpha_2, \ldots, \alpha_d) = (a_1, a_2, \ldots, a_{d+1}),$$

where

$$a_1 := \cos\alpha_1 \prod_{j=2}^{d} \sin\alpha_j, \quad a_2 := \prod_{j=1}^{d} \sin\alpha_j,$$
$$a_k := \cos\alpha_{k-1} \prod_{j=k}^{d} \sin\alpha_j, \quad k \in \{3, \ldots, d+1\}.$$

For example, if a point $\mathbf{a} \in \mathbb{S}^2$ has spherical polar coordinates (ϕ, θ), its Cartesian coordinates are $\odot(\phi, \theta) = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta)$.

For $d > 1$ the coordinate map $\odot$ defines the major colatitude to be the last one, α_d.

Besides taking the longitude modulo 2π, for $d > 1$ the coordinate map $\odot$ as given by Definition 2.3.3 is not one-to-one. In particular, for any $(\beta_1, \ldots, \beta_{d-1}) \in \mathbb{R}\times[0, \pi]^{d-2}$, for the *North pole* $\mathbf{e}_{d+1}$ we have

$$\mathbf{e}_{d+1} := (0, \ldots, 0, 1) = \odot(\beta_1, \ldots, \beta_{d-1}, 0) \tag{2.3.7}$$

and for the *South pole*, $-\mathbf{e}_{d+1}$, we have

$$-\mathbf{e}_{d+1} := (0, \ldots, 0, -1) = \odot(\beta_1, \ldots, \beta_{d-1}, \pi). \tag{2.3.8}$$

In this thesis, we identify each point of $\mathbb{S}^d$ with its Cartesian coordinate vector. To reduce ambiguity, we use bold Roman lower case letters to stand for points, eg. $\mathbf{a}$, normal Roman lower case letters to stand for Cartesian coordinates, eg. a_1, and lower case Greek letters to stand for spherical polar coordinates, eg. α_1.

The spherical polar coordinates for $\mathbb{S}^2$ can be described in terms of parallels of latitude and meridians of longitude. Here we generalize these concepts to $\mathbb{S}^d$.

Definition 2.3.4. *Let $\mathring{\mathbb{S}}^d$ denote the unit sphere $\mathbb{S}^d$ excluding the North and South poles.*

For $\mathbf{a} := \odot(\alpha_1, \ldots, \alpha_{d-1}, \alpha_d) \in \mathring{\mathbb{S}}^d$ the parallel *through $\mathbf{a}$ is*

$$\ominus(\mathbf{a}) := \{\odot(\beta_1, \ldots, \beta_{d-1}, \alpha_d) \mid (\beta_1, \ldots, \beta_{d-1}) \in [0, 2\pi) \times [0, \pi]^{d-2}\} \tag{2.3.9}$$

and the meridian *through $\mathbf{a}$ is*

$$\oplus(\mathbf{a}) := \{\odot(\alpha_1, \ldots, \alpha_{d-1}, \beta) \mid \beta \in (0, \pi)\}. \tag{2.3.10}$$

2.3.4 Stereographic projection

The equator of $\mathbb{S}^d$ lies in the subspace orthogonal to the North pole, that is $\mathbf{e}_{d+1}^{\perp}$, where

$$\mathbf{a}^{\perp} := \{\mathbf{b} \in \mathbb{R}^{d+1} \mid \mathbf{a} \cdot \mathbf{b} = 0\}. \tag{2.3.11}$$

The hyperplane $\mathbf{e}_{d+1} + \mathbf{e}_{d+1}^{\perp}$ is parallel to the equator and passes through the North pole.

The *Stereographic projection*

$$\varpi : \mathbb{R}^{d+1} \setminus (\mathbf{e}_{d+1} + \mathbf{e}_{d+1}^{\perp}) \to \mathbb{R}^d$$

based on the North pole $\mathbf{e}_{d+1}$ is defined by

$$\varpi(x_1, x_2, \ldots, x_d, x_{d+1}) := \frac{(x_1, x_2, \ldots, x_d)}{1 - x_{d+1}}. \tag{2.3.12}$$

Lemma 2.3.5. *When restricted to $\mathbb{S}^d$, the stereographic projection ϖ has the following well-known properties [79, Prop 3, p10].*

1. *The map ϖ is a one-to-one mapping from $\mathbb{S}^d \setminus \mathbf{e}_{d+1}$ onto $\mathbb{R}^d$.*

See also [45, Example 4.5, p. 19]. (Note: [123, Section 4.2, p. 112] defines the stereographic projection in the inverse sense).

2. *The map ϖ is conformal: it preserves angles. That is, at a point $\mathbf{a} \in \mathbb{S}^d$, the angle between two smooth curves B and C passing through $\mathbf{a}$ is the angle α between their respective tangent vectors. If $\mathbf{a}$ is not the North pole then when we use ϖ to map $\mathbb{S}^d \setminus \mathbf{e}_{d+1}$ onto $\mathbb{R}^d$, the curves B and C map to the curves $\varpi(B)$ and $\varpi(C)$ respectively. The curves $\varpi(B)$ and $\varpi(C)$ pass though the point $\varpi(\mathbf{a})$ with the angle between the tangent vectors corresponding to $\varpi(B)$ and $\varpi(C)$ being α, the same angle as for B and C on $\mathbb{S}^d$.*
   See also [71, P.1, pp. 14–16].
3. *The map ϖ maps generalized spheres to generalized spheres. More specifically, great and small spheres in $\mathbb{S}^d$ which do not pass through the North pole map to spheres in $\mathbb{R}^d$; and great and small spheres in $\mathbb{S}^d$ passing through the North pole map to hyperplanes in $\mathbb{R}^d$.*
   See also [71, P.7.3, p. 29].

Remarks.

1. The stereographic projection based on the South pole $-\mathbf{e}_{d+1}$ can be defined as

$$\varpi_{-\mathbf{e}_{d+1}}(x_1, x_2, \ldots, x_d, x_{d+1}) := \frac{(x_1, x_2, \ldots, x_d)}{1 + x_{d+1}}. \tag{2.3.13}$$

2. The stereographic projection $\varpi_{\mathbf{a}}$ based on any other point $\mathbf{a} \in \mathbb{S}^d$ can be defined by first rotating the sphere $\mathbb{S}^d$ in the hyperplane containing the meridian $\oplus(\mathbf{a})$, so that $\mathbf{a}$ rotates to the North pole, then using the projection ϖ.
3. Stereographic projection is much more widely known in the case of $\mathbb{S}^2$. For $\mathbb{S}^2$, see for example [22] Section 5.2, Theorems 6, 8 and 7 respectively, where $\mathbb{R}^2$ is identified with $\mathbb{C}$.

2.3.5 Spherical caps, collars, zones and spherical rectilinear regions

For $d \geqslant 1$, for any point $\mathbf{a} \in \mathbb{S}^d$ and any angle $\theta \in [0, \pi]$, the closed *spherical cap* $\mathsf{S}^d(\mathbf{a}, \theta)$ is

$$\mathsf{S}^d(\mathbf{a}, \theta) := \{\mathbf{b} \in \mathbb{S}^d \mid \mathsf{s}(\mathbf{a}, \mathbf{b}) \leqslant \theta\}, \tag{2.3.14}$$

that is the set of points of $\mathbb{S}^d$ whose spherical distance to $\mathbf{a}$ is at most θ. The angle θ is called the *spherical radius* of the cap. The notation drops the explicit dependence on d where this is understood from the context.

Note that the centre of the cap $\mathsf{S}^d(\mathbf{a}, \theta)$ lies on the sphere $\mathbb{S}^d$ and is not the same as the centre in $\mathbb{R}^{d+1}$ of the small sphere $\partial\mathsf{S}^d(\mathbf{a}, \theta)$ which is the boundary of the cap.

In this thesis will also use a notation for spherical caps based on the Euclidean distance between the centre of the cap and the boundary of the cap. If the spherical radius is θ, this Euclidean distance is $\Upsilon(\theta)$. We therefore define, for $\mathbf{a} \in \mathbb{S}^d$ and $r \in [0, 2]$,

$$\mathsf{S}^d_E(\mathbf{a}, r) := \mathsf{S}^d(\mathbf{a}, \Upsilon^{-1}(r)). \tag{2.3.15}$$

A closed spherical *collar* or annulus is the closure of the set difference between two spherical caps with the same centre and different radii.

For $d > 1$, a *zone* is a closed subset of $\mathbb{S}^d$ which can be described by

$$Z(\alpha, \beta) := \left\{\odot(\gamma_1, \ldots, \gamma_d) \in \mathbb{S}^d \mid \gamma_d \in [\alpha, \beta]\right\}, \tag{2.3.16}$$

where $0 \leqslant \alpha < \beta \leqslant \pi$.

$Z(0, \alpha)$ is a North polar cap, that is a spherical cap with centre the North pole, and $Z(\alpha, \pi)$ is a South polar cap. If $0 < \alpha < \beta < \pi$, $Z(\alpha, \beta)$ is a collar.

We note the following property of spherical distances and spherical caps on $\mathbb{S}^2$.

Lemma 2.3.6. *Let S be a spherical cap of $\mathbb{S}^2$ with centre $\mathbf{e}$ and let $\mathbf{a}$ be any point of $\mathbb{S}^2$ other than $\mathbf{e}$ or $-\mathbf{e}$. Let D be the great circle through $\mathbf{a}$, $\mathbf{e}$ and $-\mathbf{e}$. Then D*

intersects the small circle ∂S at two points $\mathbf{c}$ *and* $\mathbf{d}$*, where* $\mathbf{c}$ *is the unique point of* ∂S *furthest from* $\mathbf{a}$ *and* $\mathbf{d}$ *is the unique point of* ∂S *closest to* $\mathbf{a}$*.*

The equatorial map.

Throughout this thesis, for $d > 1$ we identify the equator of $\mathbb{S}^d$ with the unit sphere $\mathbb{S}^{d-1} \in \mathbb{R}^d$.

We define the *equatorial map* $\Pi : \mathring{\mathbb{S}}^d \to \mathbb{S}^{d-1}$, using the following construction. Take any point $\mathbf{a} = \odot(a_1, \ldots, a_d)$ of $\mathring{\mathbb{S}}^d$ and find the intersection between the equator and $\oplus(\mathbf{a})$, the meridian through $\mathbf{a}$. This is the point $\mathbf{a}' = \odot(a_1, \ldots, a_{d-1}, \frac{\pi}{2})$. Since we identify the equator of $\mathbb{S}^d$ with the unit sphere $\mathbb{S}^{d-1}$ we also identify $\mathbf{a}' \in \mathbb{S}^d$ with $\Pi\,\mathbf{a} := \odot(a_1, \ldots, a_{d-1}) \in \mathbb{S}^{d-1}$. We call $\Pi\,\mathbf{a}$ the *equatorial image* of $\mathbf{a}$ in $\mathbb{S}^{d-1}$.

By a slight abuse of notation, for any $S \subset \mathbb{S}^d$ we define the equatorial image of S to be $\Pi\,S := \Pi(S \cap \mathring{\mathbb{S}}^d)$. Thus the equatorial image of any zone of $\mathbb{S}^d$ is the whole of $\mathbb{S}^{d-1}$.

The equatorial map has the following properties.

Lemma 2.3.7. *Take any point* $\mathbf{a} \in \mathring{\mathbb{S}}^d$*, and any other point* $\mathbf{q} \in \mathring{\mathbb{S}}^d$ *where* $\mathbf{q}$ *does not lie on the great circle defined by the meridian* $\oplus(\mathbf{a})$*.*

Then $\oplus(\mathbf{a})$ *and* $\mathbf{q}$ *define a great* $\mathbb{S}^2$*, which we denote by* $G(\mathbf{a}, \mathbf{q})$*. The meridians* $\oplus(\mathbf{a})$ *and* $\oplus(\mathbf{q})$ *are also meridians of* $G(\mathbf{a}, \mathbf{q})$*, and all of the meridians of* $G(\mathbf{a}, \mathbf{q})$ *are meridians of* $\mathbb{S}^d$*.*

The equator of $G(\mathbf{a}, \mathbf{q})$ *is a great circle through* $\Pi\,\mathbf{a}$ *and* $\Pi\,\mathbf{q}$*, and is the same as the equatorial image* $\Pi\,\mathring{G}(\mathbf{a}, \mathbf{q})$*.*

Lemma 2.3.8. *Use the definitions and notation of Lemma 2.3.7. Then for any* $X \subset \mathring{\mathbb{S}}^d$ *we have*

$$\Pi\left(X \cap \mathring{G}(\mathbf{a}, \mathbf{q})\right) = \Pi\,X \cap \Pi\,\mathring{G}(\mathbf{a}, \mathbf{q}). \tag{2.3.17}$$

Lemma 2.3.9. *Consider a closed spherical cap* $\mathsf{S}(\mathbf{a}, \Phi) \subset \mathring{\mathbb{S}}^2$*, where the point* $\mathbf{a} = \odot(\alpha_1, \alpha_2)$*. The equatorial image,* $\Pi\,\mathsf{S}(\mathbf{a}, \Phi)$ *of* $\mathsf{S}(\mathbf{a}, \Phi)$ *is the same as the equatorial* *image* $\Pi\,\partial\mathsf{S}(\mathbf{a},\Phi)$ *of the boundary of* $\mathsf{S}(\mathbf{a},\Phi)$. *This image is an equatorial arc with the formula*

$$\Pi\,\mathsf{S}(\mathbf{a},\Phi) = \Pi\,\partial\mathsf{S}(\mathbf{a},\Phi) = \mathsf{S}^1(\Pi\,\mathbf{a},\phi) = \odot\left([\alpha_1-\phi,\alpha_1+\phi] \pmod{2\pi}\right), \tag{2.3.18}$$

where

$$\sin\phi = \frac{\sin\Phi}{\sin\theta}. \tag{2.3.19}$$

We now consider the equatorial image of a spherical cap in $\mathbb{S}^d$.

Lemma 2.3.10. *Consider a closed spherical cap* $\mathsf{S}(\mathbf{a},\Phi) \subset \mathring{\mathbb{S}}^d$, *where*

$$\mathbf{a} = \odot(\alpha_1,\alpha_2,\ldots,\alpha_d).$$

The equatorial image, $\Pi\,\mathsf{S}(\mathbf{a},\Phi)$ *of* $\mathsf{S}(\mathbf{a},\Phi)$ *is the same as the equatorial image* $\Pi\,\partial\mathsf{S}(\mathbf{a},\Phi)$ *of the boundary of* $\mathsf{S}(\mathbf{a},\Phi)$. *This image is a spherical cap in* $\mathbb{S}^{d-1}$ *with the formula*

$$\Pi\,\mathsf{S}(\mathbf{a},\Phi) = \Pi\,\partial\mathsf{S}(\mathbf{a},\Phi) = \mathsf{S}^{d-1}(\Pi\,\mathbf{a},\phi), \tag{2.3.20}$$

where

$$\sin\phi = \frac{\sin\Phi}{\sin\theta}. \tag{2.3.21}$$

Regions which are rectilinear in spherical polar coordinates.

To describe the recursive zonal equal area partition of Chapter 3, we need to describe regions of the form

$$\mathsf{R} = \odot\left([\tau_1,\upsilon_1]\times\ldots\times[\tau_d,\upsilon_d]\right), \tag{2.3.22}$$

where

$$\tau_1 \in [0, 2\pi), \quad \upsilon_1 \in (\tau_1, \tau_1 + 2\pi], \quad 0 \leqslant \tau_k < \upsilon_k \leqslant \pi, \quad k \in \{2, \ldots, d\}. \tag{2.3.23}$$

More specifically, we have the following definition.

Definition 2.3.11. *For the pair of d-tuples $(\tau_1, \ldots, \tau_d), (\upsilon_1, \ldots, \upsilon_d) \in \mathbb{R} \times [0, \pi]^{d-1}$ satisfying (2.3.23) we define the region*

$$\begin{aligned}\mathcal{R}\left((\tau_1, \ldots, \tau_d), (\upsilon_1, \ldots, \upsilon_d)\right) &:= \{\odot(\alpha_1, \ldots, \alpha_d) \;\mid\; \alpha_k \in [\tau_k, \upsilon_k], k \in \{1, \ldots, d\}\} \\ &= \odot\left([\tau_1, \upsilon_1] \times \ldots \times [\tau_d, \upsilon_d]\right).\end{aligned} \tag{2.3.24}$$

We define a RISC region *of $\mathbb{S}^d$ to be a region of $\mathbb{S}^d$ of the form (2.3.24) – with RISC being a near-acronym for "rectilinear in spherical polar coordinates".*

Each RISC region of $\mathbb{S}^d$ can be represented by the pair of d-tuples $(\tau_1, \ldots, \tau_d)$, $(\upsilon_1, \ldots, \upsilon_d)$.

In particular, for $d > 1$, a North polar cap of $\mathbb{S}^d$ can be described as

$$\mathcal{R}\left((0, 0, \ldots, 0, 0), (2\pi, \pi, \ldots, \pi, \upsilon_d)\right) = \odot\left([0, 2, \pi] \times [0, \pi]^{d-2} \times [0, \upsilon_d]\right),$$

and a South polar cap of $\mathbb{S}^d$ can be described as

$$\mathcal{R}\left((0, 0, \ldots, 0, \tau_d), (2\pi, \pi, \ldots, \pi, \pi)\right) = \odot\left([0, 2\pi] \times [0, \pi]^{d-2} \times [\tau_d, \pi]\right).$$

Each RISC region of $\mathbb{S}^d$ has 2^d *pseudo-vertices*, each of which is a d-tuple in spherical polar coordinates $\mathbb{R} \times [0, \pi]^{d-1}$. The term "pseudo-vertex" is used because we may have degenerate cases where the points of $\mathbb{S}^d$ corresponding to two or more of these 2^d d-tuples coincide, as must happen when $\tau_1 = 0$ and $\upsilon_1 = 2\pi$. In these degenerate cases, the corresponding point of $\mathbb{S}^d$ may be an interior point of the region, or a point where the boundary of the region is smooth. Examples are:

1. The pair $((0,0),(2\pi,\upsilon_2))$ yields the four pseudo-vertices

$$\{(0,0),(2\,\pi,0),(0,\upsilon_2),(2\,\pi,\upsilon_2)\}$$

   and the region $\mathcal{R}\left((0,0),(2\pi,\upsilon_2)\right)$ which is a North polar cap of $\mathbb{S}^2$. The pseudo-vertices $(0,0)$ and $(2\,\pi,0)$ both correspond to $\odot\left((0,0)\right)$, which is the North pole, an interior point of $\mathcal{R}\left((0,0),(2\pi,\upsilon_2)\right)$.

2. The pair $((0,0,\tau_3),(2\pi,\upsilon_2,\upsilon_3))$ yields the eight pseudo-vertices

$$\big\{(0,0,\tau_3),(2\pi,0,\tau_3),(0,\upsilon_2,\tau_3),(2\pi,\upsilon_2,\tau_3),$$
$$(0,0,\upsilon_3),(2\pi,0,\upsilon_3),(0,\upsilon_2,\upsilon_3),(2\pi,\upsilon_2,\upsilon_3)\big\}.$$

   and the region $\mathcal{R}\left((0,0,\tau_3),(2\pi,\upsilon_2,\upsilon_3)\right)$ of $\mathbb{S}^3$ which is a descendant of a polar cap in $\mathbb{S}^2$.

The following elementary relationship between RISC regions follows immediately from Definition 2.3.11.

Lemma 2.3.12. *The equatorial image of a RISC region of* $\mathbb{S}^d$ *is a region of* $\mathbb{S}^{d-1}$ *which is also RISC. Specifically, we have*

$$\Pi\,\mathcal{R}\left((\tau_1,\ldots,\tau_{d-1},\tau_d),(\upsilon_1,\ldots,\upsilon_{d-1},\upsilon_d)\right)=\mathcal{R}\left((\tau_1,\ldots,\tau_{d-1}),(\upsilon_1,\ldots,\upsilon_{d-1})\right). \tag{2.3.25}$$

The boundary $\partial\mathsf{R}$ of a RISC region $\mathsf{R}\subset\mathbb{S}^d$ consists of a set of *facets.* In general, each facet is a $d-1$ dimensional rectangular prism in spherical polar coordinates, defined by fixing one of the coordinates of R to be the high or low boundary value. For example, the *top facet* of

$$\mathsf{R}:=\mathcal{R}\left((\tau_1,\ldots,\tau_{d-1},\tau_d),(\upsilon_1,\ldots,\upsilon_{d-1},\upsilon_d)\right)$$

is

$$\mathcal{F}_{d,\uparrow}\,\mathsf{R} := \odot\big([\tau_1, \upsilon_1] \times [\tau_2, \upsilon_2] \times \ldots \times \{\tau_d\}\big), \tag{2.3.26}$$

the *bottom facet* is

$$\mathcal{F}_{d,\downarrow}\,\mathsf{R} := \odot\big([\tau_1, \upsilon_1] \times [\tau_2, \upsilon_2] \times \ldots \times \{\upsilon_d\}\big), \tag{2.3.27}$$

the *west facet* of R is

$$\mathcal{F}_{1,\downarrow}\,\mathsf{R} := \odot\big(\{\tau_1\} \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_d, \upsilon_d]\big), \tag{2.3.28}$$

and the *east facet* is

$$\mathcal{F}_{1,\uparrow}\,\mathsf{R} := \odot\big(\{\upsilon_1\} \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_d, \upsilon_d]\big). \tag{2.3.29}$$

A facet which forms part of the boundary of a RISC region is called a *boundary facet*.

If a facet has a colatitude which is fixed at 0 or π then the facet is said to be *degenerate*. A degenerate facet is not a boundary facet, but rather an artifact of the spherical polar coordinate system.

Also, if

$$\mathsf{R} = \odot\big([0, 2\pi] \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_d, \upsilon_d]\big)$$

then the east and west facets coincide and neither is a boundary facet.

If R does not intersect either the North or South poles of $\mathbb{S}^d$, then $\partial\mathsf{R}$ must contain at least a top facet and a bottom facet.

The *top boundary* of a region $\mathsf{R} \subset \mathbb{S}^d$ is

$$\partial^{\uparrow}\mathsf{R} := \partial\mathsf{R} \cap \mathcal{F}_{d,\uparrow}\,\mathsf{R}, \tag{2.3.30}$$

the *bottom boundary* is

$$\partial^{\downarrow}\mathsf{R} := \partial\mathsf{R} \cap \mathcal{F}_{d,\downarrow}\,\mathsf{R}, \tag{2.3.31}$$

and the *top and bottom boundary* is

$$\partial^{\updownarrow}\mathsf{R} := \partial^{\uparrow}\mathsf{R} \cup \partial^{\downarrow}\mathsf{R}. \tag{2.3.32}$$

A *side facet* is defined to be any boundary facet other than the top facet or the bottom facet. In the case where all facets are boundary facets, the boundary $\partial\mathsf{R}$ consists of the top and bottom facets and $2\,(d-1)$ side facets.

A side facet has either a fixed longitude, eg.

$$\mathcal{F}_{1,\uparrow}\,\mathsf{R} = \odot\left(\{\tau_1\} \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_d, \upsilon_d]\right),$$

or a fixed colatitude (other than the main colatitude), eg.

$$\mathcal{F}_{2,\downarrow}\,\mathsf{R} = \odot\left([\tau_1, \upsilon_1] \times \{\upsilon_2\} \times \ldots \times [\tau_d, \upsilon_d]\right).$$

The *side boundary* of R is

$$\partial_{\leftrightarrow}\mathsf{R} := \partial\mathsf{R} \setminus \partial^{\updownarrow}\mathsf{R}. \tag{2.3.33}$$

Lemma 2.3.13. *If R is a RISC region of $\mathbb{S}^d$ which does not intersect either the North or South poles then $\partial\,\Pi\,\mathsf{R}$ is the image under Π of the side boundary of R. In other words,*

$$\partial\,\Pi\,\mathsf{R} = \Pi\,\partial_{\leftrightarrow}\mathsf{R}. \tag{2.3.34}$$

2.3.6 The area of spheres and spherical caps

The area of a sphere.

We use σ_d to denote the Lebesgue area measure on $\mathbb{S}^d$, and we often drop the subscript where the dimension d is understood from the context.

For $d \geqslant 0$, the area of the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ is given by [112, p. 1]

$$\omega_d := \sigma_d(\mathsf{S}^d) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}. \tag{2.3.35}$$

> *Remarks.* This usage of ω_d agrees with Müller [112] and Reimer [124], but not with Landkof [93, Chapter 1, 2, p. 45] or Andrews, Askey and Roy [4, Section 9.6, p. 455], who would put ω_{d+1} where we have ω_d.

The area of a spherical cap.

It is well known ([140, (21), p. 623], [112], [94, Lemma 4.1 p. 255]) that the area of a spherical cap $\mathsf{S}(\mathbf{x}, \theta)$ of spherical radius θ and centre $\mathbf{x}$ is

$$\mathcal{V}_d(\theta) := \sigma\left(\mathsf{S}(\mathbf{x}, \theta)\right) = \omega_{d-1} \int_0^\theta \sin^{d-1} \xi \; d\xi, \tag{2.3.36}$$

independent of $\mathbf{x}$.

It can be readily seen that $\mathcal{V}_2(\theta) = 4\,\pi \sin^2\left(\frac{\theta}{2}\right)$ and $\mathcal{V}_3(\theta) = \pi\,(2\theta - \sin(2\theta))$.

In this thesis, where we use the Euclidean notation for a spherical cap $\mathsf{S}_E(\mathbf{x}, R)$, we may also need the area as a function of the Euclidean distance R. We therefore define

$$\mathcal{V}_{E,d}(R) := \sigma\left(\mathsf{S}_E(\mathbf{x}, R)\right) = \omega_{d-1} \int_0^{\Upsilon^{-1}(R)} \sin^{d-1} \xi \; d\xi, \tag{2.3.37}$$

independent of $\mathbf{x}$. It is well known that the integral above can be expressed without the use of trigonometric functions.

Lemma 2.3.14. *For $R \in (0, 2]$ and any $\mathbf{x} \in \mathbb{S}^d$, the area integral $\mathcal{V}_{E,d}(R)$ defined by (2.3.37) can be evaluated by*

$$\mathcal{V}_{E,d}(R) = \omega_{d-1} \int_0^R r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1} dr. \tag{2.3.38}$$

The area of a spherical cap can also be described using the incomplete Beta function.

Lemma 2.3.15.

$$\frac{\mathcal{V}_d(\theta)}{\omega_d} = \frac{\mathcal{V}_d(\theta)}{\mathcal{V}_d(\pi)} = \frac{B\left(\sin^2 \frac{\theta}{2}; \frac{d}{2}, \frac{d}{2}\right)}{B\left(\frac{d}{2}, \frac{d}{2}\right)} =: I\left(\sin^2 \frac{\theta}{2}; \frac{d}{2}, \frac{d}{2}\right),$$

where $B(x; a, b)$ is the incomplete Beta function [47] and $B(a, b)$ is the Beta function.

The function I of Lemma 2.3.15 is variously called the *incomplete Beta function ratio* [81, Chapter 25, p. 211], the *regularized Beta function* [160], the cumulative distribution function of the *Beta distribution*. Somewhat confusingly, some authors call I the incomplete Beta function [2, Section 26.5] [15].

The incomplete Beta function can be expressed as a hypergeometric function [2, 26.5.23], as

$$B(x; a, b) = \frac{x^a}{a} \; {}_2F_1(a, 1 - b; a + 1; x). \tag{2.3.39}$$

See [15, (1)] for a related expression. As an immediate consequence, we can express $\mathcal{V}$ in terms of a hypergeometric function, as

$$\mathcal{V}_d(\theta) = \omega_d \; \frac{2 \; x^{\frac{d}{2}}}{d} \; \frac{{}_2F_1\left(\frac{d}{2}, 1 - \frac{d}{2}; \frac{d}{2} + 1; \sin^2 \frac{\theta}{2}\right)}{B\left(\frac{d}{2}, \frac{d}{2}\right)}. \tag{2.3.40}$$

We note that since $\mathcal{V}_d$ is defined using an integral, the derivative $D\mathcal{V}_d$ is given by

$$D\mathcal{V}_d(\theta) = \omega_{d-1} \sin^{d-1} \theta, \tag{2.3.41}$$

where the limit for the derivative is defined from the right (above) at 0, and from the left (below) at π.

The following properties of the function $\mathcal{V}$ are well known.

Lemma 2.3.16. *The function $\mathcal{V}$ as defined by (2.3.36) has the following properties:*

1. *$\mathcal{V}$ is smooth.*
2. *$\mathcal{V}$ is monotonic increasing in $(0, \pi)$.*
3. *$\mathcal{V}(0) = 0$ and $\mathcal{V}$ is positive on $(0, \pi]$.*
4. *$D\mathcal{V}$ is positive and monotonic increasing in $(0, \pi/2)$.*
5. *For $\theta, h \geqslant 0$ and $\theta + h \in [0, \pi/2]$,*

$$\mathcal{V}(\theta + h) - \mathcal{V}(\theta) \in h\,[D\mathcal{V}(\theta), D\mathcal{V}(\theta + h)]\,. \tag{2.3.42}$$

6. *For $\theta, h \geqslant 0$, where $0 \leqslant \theta + h \leqslant \pi/2$,*

$$\mathcal{V}(\theta) + \mathcal{V}(h) \leqslant \mathcal{V}(\theta + h). \tag{2.3.43}$$

7. *For $\theta \in [0, \pi]$,*

$$D\mathcal{V}(\theta) = D\mathcal{V}(\pi - \theta).$$

8. *For $\theta \in [0, \pi]$,*

$$\mathcal{V}(\theta) + \mathcal{V}(\pi - \theta) = \omega_d. \tag{2.3.44}$$

The spherical radius of a cap of given area.

To determine the spherical radius θ of a cap of area v we need to solve the equation

$$\mathcal{V}_d(\theta) = v.$$

By Lemma 2.3.16, $\mathcal{V}_d$ is a smooth non-negative monotonic increasing function of θ, with $\mathcal{V}_d(0) = 0$. It therefore has an inverse, which we will call Θ_d. We then have

$$\begin{aligned} \Theta_d\big(\mathcal{V}_d(\theta)\big) &= \theta, \quad \text{for} \quad \theta \in [0, \pi], \\ \mathcal{V}_d\big(\Theta(v)\big) &= v, \quad \text{for} \quad v \in [0, \omega_d]. \end{aligned} \tag{2.3.45}$$

Lemma 2.3.17. *The function Θ_d satisfies*

$$\Theta_d(v) + \Theta_d(\omega_d - v) = \pi. \tag{2.3.46}$$

For brevity, the notation used in the remainder of this thesis usually omits the explicit dependence of $\mathcal{V}$ and Θ on d, ie. we will write $\mathcal{V}(\theta)$ for the area of a spherical cap of spherical radius θ.

Estimates.

In the estimates below we assume that $\theta \in (0, \xi]$, $\xi \in (0, \pi/2]$.

From (2.3.41) we have $D\mathcal{V}(\theta) = \omega_{d-1} \sin^{d-1} \theta$. Using the estimate (2.2.6) therefore gives us

$$D\mathcal{V}(\theta) \in [(\operatorname{sinc} \xi)^{d-1}, 1] \ \omega_{d-1}\theta^{d-1},$$

so

$$\mathcal{V}(\theta) \in [(\operatorname{sinc} \xi)^{d-1}, 1] \ \frac{\omega_{d-1}}{d} \theta^d. \tag{2.3.47}$$

If we then substitute $\Theta(v)$ for θ, we obtain for $v \in [0, \mathcal{V}(\xi)]$,

$$\Theta(v) \in [1, (\operatorname{sinc} \xi)^{\frac{1-d}{d}}] \ \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} v^{\frac{1}{d}}. \tag{2.3.48}$$

The estimates (2.3.47) and (2.3.48) are crude. There are instances where we need a sharper upper bound than that given by (2.3.47). The upper bound (2.3.47) is loose away from $\theta = 0$, especially for large d. Other estimates, eg. the estimate given by [57, Corollary 3.1 p. 467],

$$\mathcal{V}(\theta) < \frac{\omega_d}{\sqrt{2\pi}\ d} \frac{\sin^d \theta}{\cos \theta} \quad \text{for } \theta \in \left(0, \frac{\pi}{2}\right), \tag{2.3.49}$$

are more accurate for large d for θ away from $\pi/2$. See also [140, V. pp. 623–624].

The estimate below is as simple as that of [57] and is tighter for $d \geqslant 2$.

Lemma 2.3.18. *For $d \geqslant 2$ and $\theta \in [0, \pi/2)$ we have*

$$\mathcal{V}(\theta) \leqslant \frac{\omega_{d-1}}{d} \frac{\sin^d \theta}{\cos \theta}, \tag{2.3.50}$$

with equality only when $\theta = 0$.

If we combine (2.3.47) with (2.3.50) we obtain

Corollary 2.3.19. *For $d \geqslant 2$ and $\theta \in [0, \pi/2)$ we have*

$$\mathcal{V}(\theta) \in \left[\frac{1}{\operatorname{sinc} \theta}, \frac{1}{\cos \theta}\right] \frac{\omega_{d-1}}{d}\ \sin^d \theta. \tag{2.3.51}$$

The following estimate can be used to prove that (2.3.50) is tighter than (2.3.49) when $d \geqslant 2$.

Lemma 2.3.20. *For $d \geqslant 2$ we have*

$$\frac{\omega_d}{\omega_{d-1}} > \sqrt{\frac{2\pi}{d}}. \tag{2.3.52}$$

The following related estimates are also used in this thesis.

Lemma 2.3.21. *For $R \in (0, T]$, $T \in (0, 2]$ the normalized area*

$$\overset{*}{\mathcal{V}}_{E,d}(R) := \frac{\mathcal{V}_{E,d}(R)}{\omega_d} \tag{2.3.53}$$

can be estimated by

$$\overset{*}{\mathcal{V}}_{E,d}(R) \in \left[\mathsf{C}_{L,d}(T), \mathsf{C}_{H,d}\right) \frac{R^d}{d}, \tag{2.3.54}$$

where

$$\mathsf{C}_{L,d}(T) := \left(1 - \frac{T^2}{4}\right)^{\frac{d}{2}-1} \frac{\omega_{d-1}}{\omega_d}, \quad \mathsf{C}_{H,d} := \frac{\omega_{d-1}}{\omega_d}. \tag{2.3.55}$$

Lemma 2.3.22. *For $d \geqslant 2$ we have*

$$\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}} \in \left[\left(\frac{2\pi}{d}\right)^{\frac{1}{2}}, \pi^{\frac{1}{2}}\right]. \tag{2.3.56}$$

Lemma 2.3.23. *For $d \geqslant 2$ we have*

$$\left(1 + \frac{1}{\sqrt{8\pi d}} \left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d > \frac{3}{2} \tag{2.3.57}$$

and

$$\omega_{d-1} \left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + 1\right)^{d-1} > 1. \tag{2.3.58}$$

2.4 Partitions, diameter of a region

Partitions.

For the purposes of this thesis, we define an equal area partition of $\mathbb{S}^d$ in the following way.

Definition 2.4.1. *An* equal area partition *of $\mathbb{S}^d$ is a nonempty finite set P of* regions, *which are closed Lebesgue measurable subsets of $\mathbb{S}^d$ such that*

1. *the regions cover $\mathbb{S}^d$, that is*

$$\bigcup_{\mathsf{R} \in P} \mathsf{R} = \mathbb{S}^d;$$

2. *the regions have equal area, with the Lebesgue area measure σ of each $\mathsf{R} \in P$ being*

$$\sigma(\mathsf{R}) = \frac{\sigma(\mathbb{S}^d)}{|P|},$$

where $|P|$ denotes the cardinality of P; and

3. *the boundary of each region has area measure zero, that is, for each $\mathsf{R} \in P$, $\sigma(\partial\mathsf{R}) = 0$.*

Note that conditions 1 and 2 above imply that the intersection of any two regions of P has measure zero. This in turn implies that any two regions of P are either disjoint or only have boundary points in common. Condition 3 excludes pathological cases which are not of interest in this thesis.

Diameter of a region.

We also consider the Euclidean diameter of each region, defined as follows.

Definition 2.4.2. *The* diameter *of a region $\mathsf{R} \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ is*

$$\operatorname{diam} \mathsf{R} := \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in \mathsf{R}\}.$$

The following definition is used in Chapters 3 and 5.

Definition 2.4.3. *A set Z of partitions of $\mathbb{S}^d$ is said to be* diameter-bounded *with* diameter bound *$K \in \mathbb{R}_+$ if for all $P \in Z$, for each $\mathsf{R} \in P$,*

$$\operatorname{diam} \mathsf{R} \leqslant K\,|P|^{-\frac{1}{d}}.$$

2.5 Jacobi polynomials

For $\alpha, \beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}$ are a sequence of polynomials which are orthogonal on the interval $[-1, 1]$ with weight function

$$w^{(\alpha,\beta)}(x) := (1-x)^{\alpha}\ (1+x)^{\beta}, \tag{2.5.1}$$

that is, orthogonal with respect to the inner product

$$\langle f, g \rangle_{(\alpha,\beta)} := \int_{-1}^{1} f(x) g(x) w^{(\alpha,\beta)}(x) \; dx. \tag{2.5.2}$$

They are obtained by applying the Gram-Schmidt orthogonalization process to the monomials x^n [148, Section 2.1, 2.2], and normalized by defining

$$P_n^{(\alpha,\beta)}(1) := \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)} = \frac{(\alpha+1)_n}{n!}. \tag{2.5.3}$$

[148, (4.1.1), p. 58]. The last expression above uses the Pochhammer's shifted factorial [4, p. 2],

$$(x)_n := \prod_{k=0}^{n-1} (x+k) = \frac{\Gamma(x+n)}{\Gamma(x)} \qquad (-x \notin \mathbb{N}_0).$$

Here and below, we also use the normalized Jacobi polynomials $\widetilde{P}_n^{(\alpha,\beta)}$ defined by

$$\widetilde{P}_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}. \tag{2.5.4}$$

2.6 Reproducing kernel Hilbert spaces and polynomial spaces

Hilbert spaces.

A *Banach space* is a normed linear space which is complete, that is every Cauchy sequence in the norm converges in the space. A *Hilbert space* is a Banach space where the norm $\|\cdot\|$ is defined by an inner product $\langle , \rangle$, such that for a complex Hilbert space, $\|x\|^2 = \langle \overline{x}, x \rangle$, and for a real Hilbert space, $\|x\|^2 = \langle x, x \rangle$.

This thesis sometimes deals with real Hilbert spaces of functions on $\mathbb{S}^d$, notably $L^2(\mathbb{S}^d)$ with normalized inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^d} f(\mathbf{x}) g(\mathbf{x}) d\overset{*}{\sigma}(\mathbf{x}), \tag{2.6.1}$$

where

$$\overset{*}{\sigma} = \overset{*}{\sigma}_d := \frac{\sigma_d}{\omega_d}. \tag{2.6.2}$$

Certain finite dimensional Hilbert spaces are also defined and used below.

Kernels.

In this thesis, a *kernel* ϕ is a function from $\mathbb{S}^d \times \mathbb{S}^d$ to $\mathbb{R}$, which is possibly undefined on a set of measure zero.

This thesis usually deals with kernels which are a function of Euclidean or spherical distance, eg. given $u : (0, 2] \to \mathbb{R}$ we could have $\phi(\mathbf{x}, \mathbf{y}) := u(\|\mathbf{x} - \mathbf{y}\|)$; given $f : [-1, 1) \to \mathbb{R}$ we could have $\phi(\mathbf{x}, \mathbf{y}) := f(\mathbf{x} \cdot \mathbf{y})$.

> *Remarks.* Landkof [93, Chapter I, 1, p. 43] uses the term *M. Riesz kernel* to describe certain functions $k : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $k(\mathbf{x}) := u(\|\mathbf{x}\|)$ where $u : \mathbb{R}+ \to \mathbb{R}$. This thesis does not use the term "kernel" in this sense.

Reproducing kernel Hilbert spaces.

A *reproducing kernel* Hilbert space is a Hilbert space which is associated with a reproducing kernel. For kernel ϕ and $\mathbf{x} \in \mathbb{S}^d$, define $\phi_{\mathbf{x}}$ by $\phi_{\mathbf{x}}(\mathbf{y}) := \phi(\mathbf{x}, \mathbf{y})$. Then ϕ is a reproducing kernel for a Hilbert space $\mathcal{H}$ of real functions on $\mathbb{S}^d$ if and only if $\phi_{\mathbf{x}} \in \mathcal{H}$ and $\langle \phi_{\mathbf{x}}, f \rangle = f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^d$ and all $f \in \mathcal{H}$.

Thus for a reproducing kernel we also have

$$\|\phi_{\mathbf{x}}\|^2 := \phi(\mathbf{x}, \mathbf{x}) \tag{2.6.3}$$

for all $\mathbf{x} \in \mathbb{S}^d$.

Polynomial spaces on the unit sphere.

The notation used here parallels that of [124], [142] and [74].

We use $\mathbb{P}([-1,1])$ to denote the real polynomials restricted to the interval $[-1,1]$ and for any polynomial $p \in \mathbb{P}([-1,1])$ we define $\widetilde{p}$ to be the normalized polynomial $p/p(1)$.

We use $\mathring{\mathbb{P}}_t(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of homogeneous degree t, restricted to $\mathbb{S}^d$. This space is known [124, (4.1)] to have dimension

$$\mathring{\mathcal{D}}(d,t) := \dim \mathring{\mathbb{P}}_t(\mathbb{S}^d) = \binom{t+d}{d}. \tag{2.6.4}$$

We use $\mathbb{P}_t(\mathbb{S}^d)$ to denote the real polynomials on $\mathbb{R}^{d+1}$, of maximum total degree t, restricted to $\mathbb{S}^d$. This space is known [124, (4.4)] to have dimension

$$\mathcal{D}(d,t) := \dim \mathbb{P}_t(\mathbb{S}^d) = \binom{t+d}{d} + \binom{t+d-1}{d} = \frac{(2t+d)(t+d-1)!}{t!\ d!} \tag{2.6.5}$$

$$= \frac{2t+d}{t!}\ (d+1)_{t-1} = \frac{2t+d}{d!}\ (t+1)_{d-1}, \tag{2.6.6}$$

and is known [124, (4.31)] to be a reproducing kernel Hilbert space with inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^d} f(\mathbf{y}) g(\mathbf{y})\ d\sigma(\mathbf{y}) \tag{2.6.7}$$

and reproducing kernel $\Phi_t^{(d+1)}(\mathbf{x},\mathbf{y}) := \Phi_t^{(d+1)}(\mathbf{x}\cdot\mathbf{y})$, where the kernel polynomial $\Phi_t^{(d+1)}$ is defined by

$$\Phi_t^{(d+1)} := \frac{2}{\omega_d}\ \frac{(d+1)_{t-1}}{(\frac{d}{2}+1)_{t-1}}\ P_t^{(\frac{d}{2},\frac{d}{2}-1)}, \tag{2.6.8}$$

that is

$$\int_{\mathbb{S}^d} f(\mathbf{y}) \Phi_t^{(d+1)}(\mathbf{y}\cdot\mathbf{z})\ d\sigma(\mathbf{y}) = f(\mathbf{z})$$

for all $f \in \mathbb{P}_t(\mathbb{S}^d)$, $\mathbf{z} \in \mathbb{S}^d$.

Remarks. Note that the kernel polynomial is defined on $[-1, 1]$, but the kernel itself is defined on $\mathbb{S}^d \times \mathbb{S}^d$. It should be clear from the context which function is meant.

Using (2.5.3) and (2.6.6) we have

$$\Phi_t^{(d+1)}(1) = \frac{2}{\omega_d} \frac{(d+1)_{t-1}}{(\frac{d}{2}+1)_{t-1}} \frac{(\frac{d}{2}+1)_t}{t!} = \frac{1}{\omega_d} \frac{2t+d}{t!} (d+1)_{t-1} = \frac{\mathcal{D}(d,t)}{\omega_d}, \tag{2.6.9}$$

and therefore, using (2.5.4), we have

$$\Phi_t^{(d+1)} = \frac{\mathcal{D}(d,t)}{\omega_d} \widetilde{P}_t^{(\frac{d}{2},\frac{d}{2}-1)}. \tag{2.6.10}$$

2.7 Separation and packing

By the minimum distance between points of a code $X \subset \mathbb{S}^d$ we mean the minimum Euclidean distance, defined as follows.

Definition 2.7.1.

$$\operatorname{mindist}(X) := \min\{\|\mathbf{x}-\mathbf{y}\| \mid \mathbf{x},\mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\}. \tag{2.7.1}$$

The problem of maximizing the minimum distance between points of a spherical code is called the *Tammes problem*, after the botanist Pieter Merkus Lambertus Tammes, who studied the problem in his investigation into the arrangement of pores on pollen grains [150, Chapter 3, Section 1, pp. 62–71].

Well separated sequences of spherical codes.

Definition 2.7.2. *We say that a sequence $\mathcal{X}$ of $\mathbb{S}^d$ codes is* well separated *if there is a constant C_Δ such that for all $X_\ell \in \mathcal{X}$,*

$$\operatorname{mindist}(X_\ell) > \mathsf{C}_\Delta \mathcal{N}_\ell^{-\frac{1}{d}},$$

where $\mathcal{N}_\ell = |X_\ell|$.

If we use spherical rather than Euclidean distance in Definition 2.7.2, we obtain an equivalent definition, but the separation constant may be different.

Packing radius.

The *packing radius* $\operatorname{prad} X$ of a code $X \subset \mathbb{S}^d$ is the half the minimum spherical distance between codepoints of X.

$$\operatorname{prad} X := \frac{\Upsilon^{-1}(\operatorname{mindist} X)}{2}. \tag{2.7.2}$$

This is the maximum spherical radius ρ such that no two spherical caps of the set $\{\mathsf{S}(\mathbf{x}, \rho) \mid \mathbf{x} \in X\}$ have an intersection with positive area.

Definition 2.7.3. *A* saturated packing *of packing radius ρ is a packing of spherical caps of packing radius ρ such that another cap cannot be added without moving the existing caps.*

We can create a saturated packing of spherical caps with packing radius ρ by using a greedy algorithm. Place the first cap anywhere. Once i caps have been placed, let $\mathbf{x}_{i+1}$ be a point of $\mathbb{S}^d$ which is at spherical distance ρ from the union of the k caps. If there is no such point, let $\mathcal{N} := i$ and we are done. Otherwise let cap $i + 1$ have the centre $\mathbf{x}_{i+1}$.

Once we have finished the greedy algorithm, we see that no point of $\mathbb{S}^d$ is more than 2ρ from the centre of a cap, otherwise we could have added another cap of spherical radius ρ to the packing, continuing the greedy algorithm [164, p. 1091] [165, Lemma 1, p. 2112]. We therefore have the following result.

Lemma 2.7.4. *The centre points of a saturated packing of spherical caps on $\mathbb{S}^d$ with packing radius ρ are the centre points of a covering of spherical caps on $\mathbb{S}^d$ with spherical radius 2ρ. That is, if X is the spherical code whose codepoints are the* *centres of the packing caps, then*

$$\bigcup_{\mathbf{x} \in X} \mathsf{S}(\mathbf{x}, 2\rho) \supseteq \mathbb{S}^d. \tag{2.7.3}$$

Packing density.

Definition 2.7.5. *The* packing density $\operatorname{pdens} X$ *of a spherical code* $X \subset \mathbb{S}^d$ *is the ratio of area of the union of packing caps to the area of* $\mathbb{S}^d$*, that is*

$$\operatorname{pdens} X := |X| \; \frac{\mathcal{V}_d(\operatorname{prad} X)}{\omega_d}. \tag{2.7.4}$$

Voronoi cells.

Given a spherical code $X \subset \mathbb{S}^d$, the *Voronoi cell* $V_{\mathbf{x}}$ corresponding to codepoint $\mathbf{x} \in X$ consists of those points of $\mathbb{S}^d$ which are at least as close to the codepoint $\mathbf{x}$ as they are to of any of the other codepoints of X [43] [44, Vol II, pp. 37, 41] [35, 'Dirichlet regions' p. 263] [17, 'Dirichlet-Voronoi cells' pp. 243–244].

2.7.1 Bounds

There are a number of bounds associated with spherical codes and the packing of spherical caps on $\mathbb{S}^d$. The bounds can be expressed in a number of ways.

For a given number of codepoints in a packing, there are lower and upper bounds on the maximum packing radius and equivalently, lower and upper bounds on the maximum of the minimum Euclidean separation.

For a given packing radius or minimum Euclidean separation there are lower and upper bounds on the maximum number of codepoints in a packing.

Perhaps the simplest way to express these bounds is as bounds on the packing density as a function of packing radius, minimum Euclidean separation or number of codepoints.

We therefore define the maximum density function for spherical codes as

$$\operatorname{maxpdens}(d,\rho) := \max\ \{\operatorname{pdens} X \subset \mathbb{S}^d \mid \operatorname{prad} X = \rho\}. \tag{2.7.5}$$

To allow easier comparison with many of the bounds quoted in the literature, we also define $\mathcal{A}(d+1,\Delta^2)$ to be the largest size of a $\mathbb{S}^d$ code with minimum distance at least Δ [50, Section 1.3]. More formally,

$$\mathcal{A}(d+1,\Delta^2) = \max\ \{|X| \mid X \subset \mathbb{S}^d, \operatorname{mindist} X \geqslant \Delta\}. \tag{2.7.6}$$

The Chabauty-Shannon-Wyner lower bound.

The current situation for general lower bounds is considerably simpler than that for upper bounds.

The Chabauty-Shannon-Wyner (CSW) lower bound on packing density is based on the observation that for a packing to have maximum density for a given packing radius ρ, the packing must be saturated for that radius. Lemma 2.7.4 then tells us that the corresponding caps of radius 2ρ cover $\mathbb{S}^d$ and so we must have

$$\operatorname{maxpdens}(d,\rho) \geqslant \frac{\mathcal{V}_d(\rho)}{\mathcal{V}_d(2\rho)}. \tag{2.7.7}$$

See also [50, Theorem 1.6.2, pp. 21–22].

For $\rho \in (0, \frac{\pi}{4})$ the estimate (2.3.51) then gives us

$$\begin{aligned}\operatorname{maxpdens}(d,\rho) &\geqslant \frac{\cos(2\rho)}{\operatorname{sinc}\rho}\,\frac{\sin^d\rho}{\sin^d(2\rho)}\\ &\geqslant 2^{-d}\frac{\cos(2\rho)}{\operatorname{sinc}\rho\ \cos^d\rho} \geqslant 2^{-d}\cos(2\rho).\end{aligned} \tag{2.7.8}$$

There are many specific constructions which supersede the CSW lower bound [50, p. 22]. For example, the apple peeling codes of el Gamal et al. [49, p. 122] beat the CSW lower bound for at least $\mathbb{S}^2$ [66, Lemma 3], and the wrapped spherical codes of Hamkins and Zeger [65, 66] beat the CSW lower bound in general.

In fact, *any* saturated packing on $\mathbb{S}^d$ with more than 2 codepoints must do better than the CSW lower bound, because the covering caps of Lemma 2.7.4 must overlap.

Definition 2.7.6. *For the purposes of this thesis, we define the* Wyner ratio *of a spherical code X to be the ratio of the packing density* $\operatorname{pdens} X$ *to the CSW lower bound corresponding to the packing radius* $\operatorname{prad} X$*.*

Thus if the Wyner ratio of the code X is less than 1, then X does not correspond to a saturated packing, and X can in some sense be considered to be "poorly packed". A Wyner ratio of more than 1 does not necessarily mean that the corresponding packing is saturated.

Upper bounds.

Over time, the upper bounds on the maximum packing density have increased in tightness, sophistication and complexity. We have just used the naive packing bound which simply says that the packing density is at most 1. The packing arguments used in this thesis rely on this simple bound.

Below we briefly mention the more sophisticated bounds. More detailed discussion of these bounds is beyond the scope of this thesis. For some deeper overviews, more details and further references, see [65, Chapter 3], [66, Section II], [34, Chapter 1, Section 2], [50, Chapters 1 to 3], [19, Chapters 2 and 3].

The Rankin bounds.

For $\Delta \in (\sqrt{2}, 2]$, Rankin's first and second bounds [121, Theorem 1 (ii), (iii), p. 139] [34, Chapter 1, (59, 60), p. 27] [50, Section 1.4] state that

$$\mathcal{A}(d+1, \Delta^2) \leqslant \min\left(\left\lfloor \frac{\Delta^2}{\Delta^2 - 2} \right\rfloor, d+2\right). \tag{2.7.9}$$

When $\frac{\Delta^2}{\Delta^2-2} = d+2$ we have $\Delta^2 = 2\frac{d+2}{d+1}$. The spherical code with $d+2$ codepoints on $\mathbb{S}^d$ with a squared minimum distance of $2\frac{d+2}{d+1}$ consists of the vertices of a regular spherical simplex, one of the Platonic solids of $\mathbb{R}^{d+1}$. For $\mathbb{S}^2$, this is a regular tetrahedron [34, Chapter 1, p. 27] [50, Section 1.5, p. 18].

Rankin's third bound [121, Theorem 1 (iv), p. 139] [34, Chapter 1, (61), p. 27] [50, Section 1.4] states that

$$\mathcal{A}(d+1, 2) \leqslant 2d+2. \tag{2.7.10}$$

A minimum distance of $\sqrt{2}$ corresponds to a packing radius of $\frac{\pi}{4}$. The spherical code with $2d+2$ codepoints on $\mathbb{S}^d$ with packing radius $\frac{\pi}{4}$ consists of the vertices of a regular *cross polytope*, another of the Platonic solids of $\mathbb{R}^{d+1}$. For $\mathbb{S}^2$, this is a regular octahedron [34, Chapter 1, p. 27] [50, Section 1.5, pp. 18–19].

Rankin's paper also includes a theorem [121, Theorem 2, p. 193] which gives a relatively simple bound on $\mathcal{A}(d+1, \Delta^2)$ for $\Delta < \sqrt{2}$. This bound has been largely superseded by the more elaborate bounds mentioned below.

The linear programming bounds.

The linear programming bound is expressed via the following theorem which follows from [42, Theorem 4.3, p. 368].

Theorem 2.7.7. *Given $\Delta \in (0, 2)$, let $s := 1 - \frac{\Delta^2}{2}$. Let f be a real polynomial such that*

1. *$f(x) \leqslant 0$ for $x \in [-1, s]$ and*
2. *the coefficients in the Gegenbauer expansion*

$$f(x) = \sum_{k=0}^{t} f_k \widetilde{P}_k^{(\frac{d}{2}-1, \frac{d}{2}-1)}(x)$$

satisfy $f_0 > 0$ and $f_k \geqslant 0$ for $k > 0$.

Then

$$\mathcal{A}(d+1, \Delta^2) \leqslant \frac{f(1)}{f_0}. \tag{2.7.11}$$

The Levenstein bound [50, Section 2.5] [19, Section 2.5.1, pp. 18–19] is a moderately complicated bound which uses a sequence of polynomials which satisfy the linear programming criteria.

The Boyvalenkov-Danev-Boumova bound [21] [19, Section 3.7, pp. 57–61] [50, Section 2.6] improves on the Levenstein bound by using polynomials of higher degree.

The Pfender bound [118, Theorem 1.1, p2; Table 2, p. 14] improves on the linear programming bound by enlarging the function space.

The Samorodnitsky bound [136, Proposition 1.1, Corollary 1.3, p. 387] is a lower bound on the linear programming bound, which gives an indication of how much improvement may be possible with bounds of this type.

The simplex bound.

If X is a $\mathbb{S}^d$ code with $|X| > 2$ and with minimum Euclidean distance Δ, consider a regular spherical simplex T with the common Euclidean distance between the vertices being Δ. Enclose each of the $d + 1$ vertices of T in a spherical cap of spherical radius $\operatorname{prad} X = \frac{\Upsilon^{-1}(\Delta)}{2}$. Then the Fejes Tóth-Coxeter-Böröczky (simplex) bound [50, Sections 3.4, 3.6] [18, Corollary 6.4.2, p. 182] says that the packing density of X does not exceed the ratio of the area of the portion of the $d + 1$ caps which lie inside T to the area of T. This ratio is given by a rather complicated formula involving Schläfli functions, which is not repeated here because this bound is not used in this thesis.

The simplex bound was proved by Fejes Tóth [55] for $\mathbb{S}^2$, and conjectured by Coxeter [36] and proved by Böröczky [17] for $\mathbb{S}^d$ for $d > 2$.

This bound is asymptotically related to the Rogers bound [131]. See [65, Lemma 3.3, p. 31] for details.

The packing of small spherical caps in a larger cap.

In this thesis we consider bounds on the number of codepoints of a spherical code lying within a spherical cap. In order to do this we need bounds on the packing density of equal spherical caps within a larger cap. For the purposes of this thesis, we use the naive upper bound of packing density 1. This is sufficient to prove the theorems included in this thesis.

For the sake of completeness, we mention here that there are tighter and more sophisticated bounds on the number of equal spherical caps within a larger cap.

Böröczky [18, Theorem 4.4.2, Corollary 4.4.3, p. 114] proves that for $\mathbb{S}^2$, the simplex bound applies to the case of two or more equal spherical caps of spherical radius less then $\frac{\pi}{3}$ within a larger cap, giving a density less than $\frac{\pi}{\sqrt{12}}$ within the larger cap.

Bezdek, Cohn and Radin [8, Theorem 8.3, p. 9] states that for $\mathbb{S}^3$, the density of two or more equal spherical caps within a larger cap of spherical radius less than $\frac{\pi}{2}$ is less than the Roger's upper bound of $0.77963\ldots$, and conjecture [8, Conjecture 8.2, p. 9] that the bound can be improved to $\frac{\pi}{\sqrt{18}} = 0.74048\ldots$

In the general case of $\mathbb{S}^d$, the recent paper of Barg and Musin [5] can be used to find many improved bounds on the packing density of small spherical caps within a larger cap. See especially [5, Corollary 3.4, Theorem 6.1, Corollary 8.2].

Bound on the number of codepoints within a spherical cap.

Lemma 2.7.8. *Let X be a spherical code with minimum Euclidean distance $\operatorname{mindist} X$ as per Definition 2.7.1, and choose $\Delta \in (0, \operatorname{mindist} X]$.*

For $\mathbf{x} \in X$ define the counting function

$$g(R) := \left| X \cap \mathsf{S}_E(\mathbf{x}, R) \right|. \tag{2.7.12}$$

Then

$$g(R) \leqslant \frac{\overset{*}{\mathcal{V}}_E(R+\Delta)}{\overset{*}{\mathcal{V}}_E\left(\frac{\Delta}{2}\right)} \leqslant 2^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} \left(\frac{R+\Delta}{\Delta}\right)^d \tag{2.7.13}$$

and

$$g(R) \leqslant 4^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} \left(\frac{R}{\Delta}\right)^d + 1. \tag{2.7.14}$$

2.8 Communications theory and spherical codes

The multivariate standard Normal distribution $N_{d+1}(0, I)$ in $\mathbb{R}^{d+1}$ has the probability density function [10, (29.6), p. 383–384]

$$N_{d+1}(0, I)(\mathbf{x}) := (2\pi)^{-\frac{d+1}{2}} \ \exp\left(-\frac{\|\mathbf{x}\|^2}{2}\right) = \prod_{k=1}^{d+1} \frac{1}{\sqrt{2\pi}} \ \exp\left(-\frac{x_k^2}{2}\right). \tag{2.8.1}$$

In other words, it is the product of $d+1$ independent standard Normal distributions.

This distribution has the property that the probability distribution function is dependent only on the Euclidean distance from the origin. It can therefore be split into a radial component and an angular component, where the angular component is the uniform distribution over the sphere $\mathbb{S}^d$ [139, Section VII].

Shannon's model of communication [138, 139] includes a source, a message encoder, a channel encoder, a channel, a channel decoder, a message decoder and a receiver. The source may be discrete or continuous, but efficient communication between the source and the channel encoder usually demands a discrete *source code*.

Gaussian source coding begins with a source which is a random $\mathbb{R}^{d+1}$ variable with a multivariate standard Normal distribution. It involves the creation of a source code which is a finite set of points of $\mathbb{R}^{d+1}$, and an algorithm which maps $\mathbb{R}^{d+1}$ to the code in such a way as to try to minimize the expected error between the source signal and the image of the map, given the number of points [68].

Since the angular and radial components of the code are separable, it is possible to code these separately. The encoding of the angular component can be treated as the encoding of the uniform distribution over the sphere $\mathbb{S}^d$ into a spherical code. This type of encoding is also known as *spherical quantization* [65, Chapter 6] [68, 77, 78, 156, 105].

As well as being a possible component of Gaussian source coding, spherical quantization is studied for its own sake and for its application to purely angular sources.

Gaussian channel.

In Shannon's model of communication [138, 139] the codes used by the message encoder and the channel encoder are separate and can be optimized independently to match the communication environment. Shannon's *Gaussian channel* model has a signal which is a point of $\mathbb{R}^{d+1}$, a transmitter which is limited in power, which is proportional to distance of the point from the origin, and a channel which adds Gaussian noise, that is noise with an uncorrelated multivariate Normal distribution with zero mean [139, Section VII] [140, Section I, Section VI (29)].

The power limitation means that the signal is essentially limited to a ball centred on the origin. If the transmitted signal is uniformly distributed over the unit ball in $\mathbb{R}^{d+1}$ then the expected radius of the signal point approaches 1 as $d \to \infty$ [139, Section VII]. The transmitted signal can therefore be treated as a point on the sphere $\mathbb{S}^d$.

Spherical coding and decoding.

In Shannon's Gaussian channel model, the channel encoder takes a message and converts it into a sequence of codepoints, the channel adds Gaussian noise and the channel decoder attempts to map each point of the received signal back into the finite set of codepoints. Because of the noise, there is a non-zero probability that a received signal point will be mapped to the wrong codepoint [140, Section I].

In Shannon's Gaussian channel model, the codepoints of a spherical code are chosen to maximize the transmission rate at a given arbitrarily small error rate. The number of bits per point transmitted is $\log_2 \mathcal{N}$, where $\mathcal{N}$ is the number of points of the code.

In the usual spherical decoding algorithm, a received signal point is mapped to the nearest codepoint. This is called *maximum likelihood* decoding. In this case, the probability of error is the probability that a received signal point lies outside the Dirichlet-Voronoi cell of the transmitted codepoint [140, Sections I, III].

As the noise level increases, so does the spherical radius containing a given proportion of random noise vectors. To keep the error rate low, the minimum radius of each Dirichlet-Voronoi cell corresponding to the spherical code must increase as well. In other words, the packing radius must increase. A larger packing radius ultimately means a smaller number of points, but a given number of points can be arranged to maximize the packing radius [49, Section II]. Thus the study of spherical codes in communication and coding theory is related to the study of the packing of spherical caps on $\mathbb{S}^d$.

As well as efficient coding schemes, spherical coding and decoding is concerned with efficient decoding algorithms. Since spherical decoding and spherical quantization are related, an efficient spherical decoding algorithm often provides the basis for an efficient spherical quantization algorithm.

Spherical decoding differs from spherical quantization in that the signal received by the channel decoder usually does not have a uniform distribution but is concentrated towards the points of the spherical code in such a way as to try to minimize the probability that received signal point will be mapped to the wrong codepoint.

The naive version of the usual spherical decoding algorithm maps a received signal point to the nearest codepoint by determining the distance to each of the $\mathcal{N}$ codepoints. Since the number of bits per codepoint is only $\log_2(\mathcal{N})$ the performance of the naive algorithm is unacceptable. It is preferable to use an algorithm where the effort to decode a received signal point is only $\mathrm{O}\left(\log(\mathcal{N})\right)$ rather than $\mathrm{O}(\mathcal{N})$. [50, Section 11.2, pp. 390–391].

2.9 Quadrature

2.9.1 Positive weight quadrature

A *quadrature rule* $Q := (X, W)$ on $\mathbb{S}^d$ of *strength* t and *cardinality* $\mathcal{N}$ is a linear functional on the set of real-valued functions on $\mathbb{S}^d$, $(\mathbb{S}^d \to \mathbb{R})$ which is defined by a sequence X of $\mathcal{N}$ quadrature points $(\mathbf{x}_1, \ldots, \mathbf{x}_{\mathcal{N}})$ on $\mathbb{S}^d$ and a sequence W of $\mathcal{N}$ corresponding real quadrature weights $(w_1, \ldots, w_{\mathcal{N}})$, as follows

$$Q\ f := \sum_{k=1}^{\mathcal{N}} w_k\ f(\mathbf{x}_k),$$

such that, for all $p \in \mathbb{P}_t(\mathbb{S}^d)$,

$$Q\ p = \int_{\mathbb{S}^d} p(\mathbf{y})\ d\sigma(\mathbf{y}).$$

A *positive weight* quadrature rule has all weights positive.

Recent papers which involve positive weight quadrature on the sphere include [108, 169, 41, 25, 75, 113, 101].

2.9.2 Spherical designs

A *spherical t-design* on $\mathbb{S}^d$ is an equal weight quadrature rule of strength t. There are many equivalent definitions of spherical designs [42, Definition 5.1, p. 371] [103, Definition 4.1, p. 340] [137] [19, Section 2.7, pp. 24–26] [144, II, pp. 253–254].

Since a spherical t-design has a strength t of at least zero, the weight for a $\mathcal{N}$ point t-design on $\mathbb{S}^d$ must be $\frac{\omega_d}{\mathcal{N}}$. In other words, with this weight, any $\mathbb{S}^d$ code is at least a spherical 0-design. Therefore the generic term *spherical design* usually means a spherical t-design with $t > 0$.

Well separated sequences of spherical designs.

The following remarks paraphrase [73, Section 5].

It has been known since the original paper of Delsarte, et al. [42] that any disjoint union of spherical t-designs is a spherical t-design. This leads to the following classification of spherical t-designs.

- Compound. A disjoint union of two or more spherical t-designs.
- Degenerate. A quadrature rule of strength t with less than $\mathcal{N}$ points, where each weight is a positive integer multiple of $\frac{\omega_d}{\mathcal{N}}$, can be considered to be a degenerate $\mathcal{N}$ point spherical t-design with a number of co-incident points.
- Simple. Neither compound nor degenerate.

We can use this classification to examine whether a particular sequence of spherical designs can be well separated as per Definition 2.7.2. We first note that any sequence of spherical designs with a degenerate member is not well separated.

It is also quite easy to construct an infinite sequence of non-degenerate spherical designs on $\mathbb{S}^d$ which is not well separated. This is because a compound spherical design can have points which are arbitrarily close together.

Given any compound spherical t-design, it is easy to construct an infinite sequence of spherical t-designs where the number of points remains constant, but the minimum distance approaches zero. This can be done by rotating one component of the compound spherical design with respect to the other components, in such a way that two of the points approach each other. Specific example of starting points for such a sequence are any compound spherical 1-design consisting of two pairs of opposite points, and any compound spherical 3-design consisting of the vertices of two cubes.

Using a similar principle of construction, it is possible to construct an infinite sequence of compound spherical designs, with increasing strength, where the minimum distance between codepoints decreases arbitrarily rapidly, and which is therefore not well separated.

2.10 Polynomial interpolation, fundamental systems

The following definitions are based on [162, Sections 1,2] [124, Sections 5.3] but with notation modified to match this thesis.

Definition 2.10.1. *A* fundamental system X *of degree* t *on* $\mathbb{S}^d$ *is a set of* $\mathcal{D}(d,t)$ *points on* $\mathbb{S}^d$ *such that the zero polynomial is the only polynomial of* $\mathbb{P}_t(\mathbb{S}^d)$ *which is zero at all points of* X*.*

Fundamental systems exist for all degrees and dimensions. For proof, see [124, Theorem 5.14, pp. 126–127].

Let $L^\infty(\mathbb{S}^d)$ denote the space of bounded real functions on $\mathbb{S}^d$, that is those $f : \mathbb{S}^d \to \mathbb{R}$ such that $\|f\|_\infty < \infty$.

Definition 2.10.2. *For a given fundamental system* X *of degree* t *on* $\mathbb{S}^d$*, the polynomial interpolation operator* Λ_X *is a projection which maps* $L^\infty(\mathbb{S}^d)$ *to the polynomial space* $\mathbb{P}_t(\mathbb{S}^d)$*. This operator satisfies* $\Lambda_X f(\mathbf{x}) = f(\mathbf{x})$ *for all* $\mathbf{x} \in X$*. See also [124, Definition 5.10, p. 130].*

Fundamental spherical designs.

Define a *fundamental spherical design* to be a fundamental system of degree t which is also a spherical t-design as per Section 2.9.2.

For $d > 1$ it is currently unknown whether spherical t-designs of cardinality $\mathbb{P}_t(\mathbb{S}^d)$ exist for all strengths t, let alone whether there are fundamental spherical designs for all strengths. See Section 5.3 and also [31].

2.11 Energy, weak-star convergence

2.11.1 Weak convergence of measures

Definition 2.11.1. *A sequence of measures* $(\nu_1, \nu_2, \ldots)$ *on a compact metric space* S converges weakly *to the measure* ν *if and only if*

$$\int_S f(\mathbf{x})\ d\nu_\ell(\mathbf{x}) \to \int_S f(\mathbf{x})\ d\nu(\mathbf{x})$$

as $\ell \to \infty$ *for all continuous* f *[9, 11].*

Definition 2.11.2. *A spherical code $X \subset \mathbb{S}^d$ defines a* normalized counting measure *via equal weight quadrature,*

$$\mathring{\sigma}_X(A) := \frac{|X \cap A|}{|X|}, \tag{2.11.1}$$

so that

$$\int_{\mathbb{S}^d} f(\mathbf{x})\ d\mathring{\sigma}_X(\mathbf{x}) = \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x})$$

for all continuous $f : \mathbb{S}^d \to \mathbb{R}$, and we can use this as the definition of a linear functional defined on $L^\infty(\mathbb{S}^d)$, the bounded real functions on $\mathbb{S}^d$

For a fixed sequence $\mathcal{X}$ of spherical codes, we often use the abbreviation $\mathring{\sigma}_\ell$ to mean $\mathring{\sigma}_{X_\ell}$.

Definition 2.11.3. *We say that the sequence $\mathcal{X} = (X_\ell, \ell \in \mathbb{N})$ of $\mathbb{S}^d$ codes is* weak-star convergent *if the corresponding sequence of normalized counting measures $(\mathring{\sigma}_{X_\ell}, \ell \in \mathbb{N})$ defined by Definition 2.11.2 converges weakly to $\mathring{\sigma}$, the normalized Lebesgue area measure on $\mathbb{S}^d$.*

A weak-star convergent sequence of codes has the following useful property.

Lemma 2.11.4. *If $\mathcal{X} = (X_\ell, \ell \in \mathbb{N})$ is a weak-star convergent sequence of $\mathbb{S}^d$ codes then the cardinality of the point sets of $\mathcal{X}$ diverges to infinity. That is, for any cardinality $N_0 > 0$ there is an index $L_0 > 0$ such that*

$$\mathcal{N}_\ell > N_0 \quad \textit{for all } \ell \geqslant L_0,$$

where $\mathcal{N}_\ell$ is the cardinality of X_ℓ.

Thus for a given weak-star convergent sequence $\mathcal{X}$ of $\mathbb{S}^d$ codes, the function

$$\mathcal{L}(N) := \min\{L \in \mathbb{N} \mid \mathcal{N}_\ell > N \text{ for all } \ell \geqslant L\} \tag{2.11.2}$$

is well defined and finite for each $N \in \mathbb{N}$.

2.11.2 Spherical cap discrepancy

Definition 2.11.5. *The normalized* spherical cap discrepancy *of a spherical code is the supremum over all spherical caps of the difference between the normalized area of the cap and the proportion of codepoints which lie in the cap. In other words,*

$$\operatorname{disc}(X) := \sup_{\mathbf{y}\in\mathbb{S}^d,\theta\in(0,\pi]} |\overset{*}{\sigma} - \overset{*}{\sigma}_X| \ \mathsf{S}(\mathbf{y},\theta). \tag{2.11.3}$$

Remarks. This is the normalized spherical cap discrepancy. Most authors use the unnormalized discrepancy [6, Theorem 24D, p. 182] [30, Section 2.5], which is larger by a factor of $|X|$.

2.11.3 Weak-star convergence and normalized spherical cap discrepancy

It has long been known that there is a relationship between weak convergence of a sequence of measures and uniform convergence of the same sequence on a subclass of sets or a subclass of functions.

A paper by R. Ranga Rao [122] was one of the first systematic expositions of this relationship. His Theorem 4.1 [122, p. 665] states that given a measure μ on $\mathbb{R}^{d+1}$ such that $\mu\mathcal{L}^{-1}$ is continuous for every linear function $\mathcal{L}$ on $\mathbb{R}^{d+1}$, a sequence of measures converges weakly to μ if and only if it converges to μ in certain discrepancies defined on half spaces.

This theorem can be used to show that a sequence of $\mathbb{S}^d$ codes is weak-star convergent if and only if it is convergent to zero in normalized spherical cap discrepancy.

Lemma 2.11.6. *A sequence $\mathcal{X}$ of $\mathbb{S}^d$ codes is weak-star convergent if and only if the corresponding sequence of normalized spherical cap discrepancies converges to zero.*

Remarks.

Lemma 2.11.6 is well known. Brauchart proves it in another way in his Diplomarbeit [23], by appealing to Grabner's [63] Erdös-Turán inequality on the sphere.

Damelin and Grabner [39, Remark 4, p. 236] use the term *asymptotically equidistributed* to describe a sequence of spherical codes whose corresponding sequence of normalized spherical cap discrepancies converges to zero. Thus Lemma 2.11.6 can be restated as: "A sequence of spherical codes is weak-star convergent if and only if it is asymptotically equidistributed". In the remainder of this thesis, we use these two equivalent properties interchangeably.

Billingsley and Topsøe [12] generalized and extended the results of Ranga Rao. See especially Theorem 2, [12, p. 2].

The theory of the relationship between weak convergence and uniform convergence is not restricted to the sphere or even to finite dimensional manifolds. Both [122] and [12] treat measures on separable metric spaces in general, and applications include laws of large numbers and Glivenko-Cantelli theory [122, p660].

Kuipers and Niederreiter prove Lemma 2.11.6 in the restricted case where the sequence of $\mathbb{S}^d$ codes is taken from a sequence of points on $\mathbb{S}^d$ by adding one point at a time, but do so in the more general context of compact spaces and continuity sets. See [90, Theorem 1.2, Example 1.3, p. 175].

Dudley's book on uniform central limit theorems [46] contains a chapter on Vapnik–Černovenkis (V-C) combinatorics. In particular, the example [46, Section 4.2 Example I, p. 140] uses the polynomials of degree at most k on $\mathbb{R}^d$ to create a V-C class which contains all ellipsoids in $\mathbb{R}^d$. This can be used as a basis for an example on $\mathbb{S}^d$.

Let L be the space of polynomials of degree at most 1 on $\mathbb{R}^{d+1}$, restricted to $\mathbb{S}^d$, ie. the linear functions. L is a vector space of dimension $d+2$, and so $\mathrm{pos}(L)$ is a V-C class. But $\mathrm{pos}(L)$ is just the set of all (open) spherical caps. Also, by [46, Theorem 4.2.1, p. 139], since L contains the constants, we have (using Dudley's notation [46, Section 4.2])

$$S(\mathrm{pos}(L)) = S(\mathrm{nn}(L)) = S(U(L)) = d + 2.$$

Dudley's notes [46, Section 4.2, p. 167] say that Theorem 4.2.1 in the case of linear function on $\mathbb{R}^m$ and $f = 0$ is known as Radon's theorem, so the equivalent for $\mathbb{S}^d$ could be called Radon's theorem on the sphere.

2.11.4 Energy functionals

For a measure ν on the compact metric space S and a point $\mathbf{x} \in S$, define the *punctured measure* $\nu[\mathbf{x}]$ by

$$\nu[\mathbf{x}](A) := \nu\left(A \setminus \{\mathbf{x}\}\right)$$

for all measurable $A \subset S$. Then, for example, for the $\mathbb{S}^d$ code X_ℓ with $\mathcal{N}_\ell$ points, and the point $\mathbf{x}_{\ell,k} \in X_\ell$, we have

$$\int_{\mathbb{S}^d} f(\mathbf{y})\ d\overset{*}{\sigma}_{X_\ell}[\mathbf{x}_{\ell,k}](\mathbf{y}) = \frac{1}{\mathcal{N}_\ell} \sum_{\substack{j=1 \\ j \neq k}}^{\mathcal{N}_\ell} f(\mathbf{x}_{\ell,j}). \tag{2.11.4}$$

for all functions f which are defined at the points of evaluation. Later, when we discuss the weak-star convergence of sequences of punctured counting measures such as $\overset{*}{\sigma}_{X_\ell}[\mathbf{x}_{\ell,k}]$, we will need to restrict the associated linear functional (2.11.4) to the space of continuous functions on $\mathbb{S}^d$.

Given a potential, usually a decreasing function of distance, we can define the energy of a spherical code. For example, the Riesz s energy is defined using the Riesz r^{-s} potential [39, 24].

Definition 2.11.7. *For a measure ν defined on $\mathbb{S}^d$, for a real potential u defined on $[0,2]$ define*

$$\mathcal{I}(\nu)\ u := \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} u(\|\mathbf{x} - \mathbf{y}\|)\ d\nu(\mathbf{y})\ d\nu(\mathbf{x}),$$

and for a real potential u *defined on* $(0,2]$ *define*

$$\mathrm{E}(\nu)\ u := \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} u(\|\mathbf{x}-\mathbf{y}\|)\ d\nu[\mathbf{x}](\mathbf{y})\ d\nu(\mathbf{x}).$$

For the normalized Lebesgue area measure $\overset{*}{\sigma}$ *defined on* $\mathbb{S}^d$ *we define the abbreviations*

$$\mathcal{I} := \mathcal{I}(\overset{*}{\sigma}), \quad \mathrm{E} := \mathrm{E}(\overset{*}{\sigma}).$$

For the normalized counting measure $\overset{*}{\sigma}_{X_\ell}$ *defined by (2.11.1) for a sequence* $\mathcal{X} = (X_\ell, \ell \in \mathbb{N}))$ *on* $\mathbb{S}^d$*, we define the abbreviations*

$$\mathcal{I}_\ell(\mathcal{X}) := \mathcal{I}(X_\ell), := \mathcal{I}(\overset{*}{\sigma}_{X_\ell}), \quad \mathrm{E}_\ell(\mathcal{X}) := \mathrm{E}(X_\ell) := \mathrm{E}(\overset{*}{\sigma}_{X_\ell}).$$

The *Riesz potential* for the exponent s is

$$\mathsf{U}_s(r) := r^{-s}, \quad 0 < s < d.$$

For $s > 0$, since $\mathsf{U}_s(r)$ diverges to $+\infty$ as $r \to 0$, we extend the definition of U_s to $[0,2]$, by setting $\mathsf{U}_s(0) := +\infty$.

For a sequence of codes $\mathcal{X}$ on $\mathbb{S}^d$, we therefore have

$$\mathcal{I}\,\mathsf{U}_s = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \|\mathbf{x}-\mathbf{y}\|^{-s}\ d\overset{*}{\sigma}(\mathbf{x})\ d\overset{*}{\sigma}(\mathbf{y}) \tag{2.11.5}$$

and

$$\mathrm{E}_\ell(\mathcal{X})\mathsf{U}_s = \frac{1}{\mathcal{N}_\ell^2}\sum_{k=1}^{\mathcal{N}_\ell}\sum_{\substack{j=1\\ j\neq k}}^{\mathcal{N}_\ell} \|\mathbf{x}_{\ell,k}-\mathbf{x}_{\ell,j}\|^{-s}. \tag{2.11.6}$$

It is well known that for $0 < s < d$ an increasing sequence of minimal s energy $\mathbb{S}^d$ codes is asymptotically equidistributed [39, p. 236]. The result in terms of weak star convergence goes back at least as far as Landkof [93, Chapter II, Section 1.1, pp. 131–133, Section 3.12, p. 160–162]. See also Frostman [61, Section 25, pp. 46–48, Section 3, pp. 12–16]. For $d-1 < s < d$ such a sequence is also well separated [89]. This also holds for $s = d-2$ when $d \geqslant 3$ [40, Theorem 3.5, p. 853].

The problem of minimizing the Coulomb (Riesz 1) energy of a $\mathbb{S}^2$ code is called the *Thomson problem*, after the physicist Joseph John Thomson, who studied a related but different arrangement of point charges in one of his investigations into atomic structure [155, p. 255].

We now list a few results on integrals related to energy functionals, all of which are well known.

Lemma 2.11.8. *For $R \in (0,2]$ and any $\mathbf{x} \in \mathbb{S}^d$, for any potential $u : (0,2] \to \mathbb{R}$, the single integral*

$$\mathcal{J}_d(\mathbf{x}; R)u := \int_{\|\mathbf{x}-\mathbf{y}\| \leqslant R} u(\|\mathbf{x}-\mathbf{y}\|)\ d\overset{*}{\sigma}(\mathbf{y}) \tag{2.11.7}$$

can be evaluated by

$$\mathcal{J}_d(\mathbf{x}; R)\ u = \mathcal{J}_d(R)\ u := \frac{\omega_{d-1}}{\omega_d} \int_0^R u(r)\ r^{d-1}\ \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1}\ dr, \tag{2.11.8}$$

which is independent of $\mathbf{x}$.

Corollary 2.11.9. *For any potential u $u : (0,2] \to \mathbb{R}$, the double integral*

$$\mathcal{I}\, u = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} u(\|\mathbf{x}-\mathbf{y}\|)\ d\overset{*}{\sigma}(\mathbf{y})\ d\overset{*}{\sigma}(\mathbf{x})$$

can be evaluated by

$$\mathcal{I}\, u = \mathcal{J}_d(2)\ u,$$

where $\mathcal{J}_d$ is defined by (2.11.8).

Corollary 2.11.10. *For $s \in (0, d)$, $R \in (0, T]$, $T \in (0, 2]$ and any $\mathbf{x} \in \mathbb{S}^d$, the integral*

$$\mathcal{J}_d(\mathbf{x}; R)\mathsf{U}_s = \int_{\|\mathbf{x}-\mathbf{y}\| \leqslant R} \|\mathbf{x} - \mathbf{y}\|^{-s} \; d\sigma^*_d(\mathbf{y}) \tag{2.11.9}$$

can be evaluated by

$$\mathcal{J}_d(\mathbf{x}; R)\mathsf{U}_s = \mathcal{J}_d(R)\mathsf{U}_s = \frac{\omega_{d-1}}{\omega_d} \int_0^R r^{d-s-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1} \; dr, \tag{2.11.10}$$

and can be estimated by

$$\mathcal{J}_d(R) \; \mathsf{U}_s \in \left[\mathsf{C}_{L,d}(T), \mathsf{C}_{H,d}\right) \; \frac{R^{d-s}}{d-s}, \tag{2.11.11}$$

where $\mathsf{C}_{L,d}$ and $\mathsf{C}_{H,d}$ are defined by (2.3.55).

2.12 Proofs of lemmas

2.12.1 Trigonometric functions and the Gamma function

Proof of Lemma 2.2.1.

For $\theta, \phi \in \mathbb{R}$ we have, by the well known addition formulae,

$$\begin{aligned}
\sin(\theta + \phi) - \sin\theta = \sin\phi \; \cos\theta + \cos\phi \; \sin\theta - \sin\theta &= \sin\phi \; \cos\theta + (\cos\phi - 1) \; \sin\theta \\
&= 2 \; \sin\frac{\phi}{2} \; \cos\frac{\phi}{2} \; \cos\theta - 2 \; \sin^2\frac{\phi}{2} \; \sin\theta \\
&= 2 \; \sin\frac{\phi}{2} \left(\cos\frac{\phi}{2} \; \cos\theta - \sin\frac{\phi}{2} \; \sin\theta\right) \\
&= 2 \; \sin\frac{\phi}{2} \; \cos\left(\theta + \frac{\phi}{2}\right).
\end{aligned}$$

Therefore for $\phi \in (0, \pi]$, $\theta \in (0, \pi/2 - \phi/2]$ we have $\sin(\theta + \phi) > \sin\theta > 0$. □

Proof of Lemma 2.2.3.

The result follows from [4, Corollary 1.2.6, Theorem 1.2.7, p. 13]. Following [4, p. 13], within this proof we use the notation

$$\psi(x) := \frac{d}{dx} \log \Gamma(x).$$

By [4, (1.2.16)] and [2, 6.3.3] we have

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2 = -1.96351\ldots.$$

By [4, (1.2.16)] we therefore have

$$\psi\left(\frac{3}{2}\right) = 2 - \gamma - 2\log 2 = 2 - 1.96351\ldots = 0.0036\ldots > 0.$$

We could have used [2, 6.3.4] directly to obtain the same result.

For the general case $x > \frac{3}{2}$ we use the log-convexity of the Gamma function (2.2.10). □

Proof of Lemma 2.2.4.

We treat the case $x = 1$ first. From (2.2.8) we see that this gives equality.

The more general case follows from (2.2.7), (2.2.9) and the Legendre duplication formula (2.2.11).

These give us

$$\begin{aligned}\Gamma(2x) &= \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\;\;\Gamma\left(x+\frac{1}{2}\right) = \frac{2^{2x-2}}{\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}\,\Gamma(x)\;\;\Gamma\left(x+\frac{1}{2}\right)\\ &= 4^{x-1}\,\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\,\Gamma(x) > 4^{x-1}\,\Gamma(x)\end{aligned}$$

for $x > 1$, where the final inequality results from (2.2.12). □

Proof of Lemma 2.2.5.

The result follows from the log-convexity of the Gamma function (2.2.10), since

$$x = \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)},$$

with the log-convexity giving us

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} > \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} > 0.$$

$\square$

Proof of Lemma 2.2.6.

From (2.2.8) and the log-convexity of the Gamma function (2.2.10), we have $\log\Gamma(1) = \log\Gamma(2) = 0$ and $\log\Gamma(x+1) < 0$ for $x \in (0,1)$.

Using (2.2.7), we also have

$$\log(x+1)^x = x\log(x+1) > 0 \quad \text{for } x > 0.$$

Therefore $\Gamma(x+1) < (x+1)^x$ for $x \in (0,1]$, and so

$$\Gamma(x+1) = x\,\Gamma(x) < x\; x^{x-1} = x^x \quad \text{for } x \in (1,2].$$

If we have x such that $\Gamma(x+1) < x^x$ then

$$\Gamma(x+2) = (x+1)\,\Gamma(x+1) < (x+1)\; x^x < (x+1)^{x+1}.$$

Therefore by induction, $\Gamma(x+1) < x^x$ for $x > 1$. Finally, $\Gamma(1+1) = 1^1$. $\square$

2.12.2 The geometry of the unit sphere $\mathbb{S}^d$

Euclidean and spherical distances

Proof of Lemma 2.3.2.

These results are well known, but we include a proof for completeness. In order,

1. For a proof for $\mathbb{S}^2$ using the Frenet equation, see [115, Example 5.8, pp. 228–229]. For an elementary proof for $\mathbb{S}^2$ using stereographic projection, see [102].
2. The result is a consequence of property 1 and the length-minimizing properties of geodesics. For a proof for $\mathbb{S}^2$, see [115, Example 5.7, pp. 346–347].
3. It is well known that any two points $\mathbf{a}, \mathbf{b} \in \mathbb{S}^d$ define at least one great circle. If $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a}$ and $\mathbf{b}$ are not antipodal, the great circle is unique. If $\mathbf{a} = \mathbf{b}$, the spherical distance and geodesic arc length are both zero. If the two points are antipodal, the spherical distance and geodesic arc length are both π.

   The arc length of an arc of a great circle is the angle at the centre $\mathbf{o}$ of the circle. Up to π, this angle is

   $$\angle\mathbf{aob} = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}) = \mathsf{s}(\mathbf{a}, \mathbf{b}).$$

   See also [115, Example 5.7, pp. 346–347].
4. Abbreviating $\mathsf{s}(\mathbf{a}, \mathbf{b})$ to s and $\|\mathbf{a} - \mathbf{b}\|$ to e, we have

   $$e^2 = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = 2 - 2\cos s,$$

   so

   $$e = \sqrt{2 - 2\cos s} = \Upsilon(s).$$

   Using the half angle formula for $\cos$, we also have

   $$e^2 = 2 - 2\cos s = 2 - 2\left(\cos^2 \frac{s}{2} - \sin^2 \frac{s}{2}\right) = 2 - 2\cos^2 \frac{s}{2} + 2\sin^2 \frac{s}{2} = 4\sin^2 \frac{s}{2},$$

   so

   $$\Upsilon(s) = 2\sin \frac{s}{2}.$$

5. Using property 4, we see that the function Υ is differentiable in $[0, \pi]$, with

$$D\,\Upsilon(s) = \frac{\partial}{\partial s}\left(2\sin\frac{s}{2}\right) = \cos\frac{s}{2} > 0 \quad \text{when} \quad s \in [0, \pi).$$

6. We use the addition theorem for the sine function. For $\alpha, \beta \in (0, \pi]$,

$$\begin{aligned}\Upsilon(\alpha+\beta) &= 2\ \sin\left(\frac{\alpha+\beta}{2}\right) = 2\ \sin\frac{\alpha}{2}\ \cos\frac{\beta}{2} + 2\ \sin\frac{\beta}{2})\ \cos\frac{\alpha}{2}\\ &< 2\ \sin\frac{\alpha}{2} + 2\ \sin\frac{\beta}{2} = \Upsilon(\alpha) + \Upsilon(\beta),\end{aligned}$$

since $\sin\frac{\alpha}{2}$ and $\sin\frac{\beta}{2}$ are both positive.

The equality when $\alpha = 0$ or $\beta = 0$ occurs because $\sin(0) = 0$.

7. For $a, b, a+b \in [0, 2]$ apply (2.3.4) with $\alpha = \Upsilon^{-1}(a)$, $\beta = \Upsilon^{-1}(b)$, to obtain

$$\Upsilon\left(\Upsilon^{-1}(a) + \Upsilon^{-1}(b)\right) \leqslant a + b,$$

which implies that

$$\Upsilon^{-1}(a) + \Upsilon^{-1}(b) \leqslant \Upsilon(a+b),$$

with equality only when $a = 0$ or $b = 0$.

8. Abbreviating $\mathsf{s}(\mathbf{a}, \mathbf{b})$ to s and $\|\mathbf{a} - \mathbf{b}\|$ to e, and using (2.3.3) and (2.3.2), we have

$$e = 2\sin\frac{s}{2} < s, \quad \text{when} \quad s > 0.$$

9. Abbreviating $\mathsf{s}(\mathbf{a}, \mathbf{b})$ to s and $\|\mathbf{a} - \mathbf{b}\|$ to e, using (2.3.3) and (2.3.2), and expanding in Taylor series, we have

$$\frac{e}{s} = \frac{\Upsilon(s)}{s} = \frac{2\sin\frac{s}{2}}{s} = \frac{s - \frac{s^3}{2^2\,3!} + \dots}{s} = 1 - \frac{s^2}{2^2\,3!} + \dots \to 1 \quad \text{as} \quad \mathbf{a} \to \mathbf{b}.$$

$\square$

Spherical caps, collars, zones and spherical rectilinear regions

Proof of Lemma 2.3.6.

Use stereographic projection from the point $-\mathbf{a}$. The point $\mathbf{a}$ projects to the origin of $\mathbb{R}^2$, the small circle ∂S projects to a circle $\partial S'$ and the points $\mathbf{e}$ and $-\mathbf{e}$ project to the points $\mathbf{e}'$ and $\mathbf{e}''$ respectively. (The point $\mathbf{e}'$ is in general not the centre of the circle $\partial S'$, but this does not matter for the purposes of this proof.)

The great circle D projects to the line D' which passes through $\mathbf{e}'$, the origin and $\mathbf{e}''$. By symmetry, the line D' passes through the centre of the circle $\partial S'$.

The line D' intersects $\partial S'$ at the points $\mathbf{c}'$ and $\mathbf{d}'$. The circle $\partial S'$ does not have the origin as its centre, since this would imply that $\mathbf{a}$ is either $\mathbf{e}$ or $-\mathbf{e}$ and we have excluded this possibility. Therefore the points $\mathbf{c}'$ and $\mathbf{d}'$ are at different distances to the origin.

Let us take $\mathbf{d}'$ to be the closer of these two points to the origin. The point $\mathbf{d}'$ is the image under stereographic projection of the point $\mathbf{d}$ which is one of the two points of intersection of ∂S and D.

Now consider the circle C' about the origin with radius $\|\mathbf{d}'\|$. The circle C' is tangent to $\partial S'$ at $\mathbf{d}'$ because the line D' passes through the origin, $\mathbf{d}'$ and the centre of $\partial S'$ [32, Proposition 84.1, p. 105].

The preimage of C' under stereographic projection is a small circle C which is the boundary of the spherical cap $\mathsf{S}(\mathbf{a}, \mathsf{s}(\mathbf{a}, \mathbf{d}))$. This cap contains no point of ∂S other than $\mathbf{d}$. Thus $\mathbf{d}$ is the unique point of ∂S which is closest to $\mathbf{a}$.

A similar argument shows that $\mathbf{c}$ is the unique point of ∂S which is furthest from $\mathbf{a}$. $\square$

Proof of Lemma 2.3.7.

Consider the embedding of $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$. The poles of $\mathbb{S}^d$ and the point $\mathbf{a}$ define a 2-plane $U_{\mathbf{a}}$ which intersects $\mathbb{S}^d$ in a great circle $C_{\mathbf{a}}$ which contains the meridian $\Phi(\mathbf{a})$ and the poles. Since the point $\mathbf{q}$ is not on $C_{\mathbf{a}}$, it does not lie in the 2-plane $U_{\mathbf{a}}$. Therefore $\Phi(\mathbf{a})$ and $\mathbf{q}$ define a 3 dimensional subspace $T(\mathbf{a}, \mathbf{q})$ of $\mathbb{R}^d$. In this subspace,

$\oplus(\mathbf{a})$ and $\mathbf{q}$ lie on a unit 2-sphere, which we have called $G(\mathbf{a},\mathbf{q})$. Since $G(\mathbf{a},\mathbf{q})$ contains $C_{\mathbf{a}}$ which contains the meridian $\oplus(\mathbf{a})$ it therefore contains the poles of $\mathbb{S}^d$ and the point $\Pi\,\mathbf{a}$.

The subspace $T(\mathbf{a},\mathbf{q})$ also contains the 2-plane $U_{\mathbf{q}}$ defined by the poles of $\mathbb{S}^d$ and the point $\mathbf{q}$. The 2-plane $U_{\mathbf{q}}$ intersects $\mathbb{S}^d$ in the great circle $C_{\mathbf{q}}$ which contains the poles and the point $\mathbf{q}$. Since $U_{\mathbf{q}}$ is a subspace of $T(\mathbf{a},\mathbf{q})$ we must have $C_{\mathbf{q}} \subset G(\mathbf{a},\mathbf{q})$. This implies that $G(\mathbf{a},\mathbf{q})$ contains the meridian $\oplus(\mathbf{q})$ and the point $\Pi\,\mathbf{q}$.

Any point $\mathbf{x} \in \mathring{G}(\mathbf{a},\mathbf{q})$ is also a point of $\mathring{\mathbb{S}}^d$. The point $\mathbf{x}$ and the poles define a 2-plane $U_{\mathbf{x}}$ which contains the great circle $C_{\mathbf{x}} \subset \mathbb{S}^d$ which contains the meridian $\oplus(\mathbf{x})$. The meridian $\oplus(\mathbf{x})$ is evidently contained in $\mathring{G}(\mathbf{a},\mathbf{q})$ and within $G(\mathbf{a},\mathbf{q})$ is an arc of a great circle between the poles extending from the North pole to the South pole but not including either pole.

The points $\Pi\,\mathbf{a}$ and $\Pi\,\mathbf{q}$ are each at spherical distance $\frac{\pi}{2}$ from both of the poles of $\mathbb{S}^d$ and so they both lie on the equator of $G(\mathbf{a},\mathbf{q})$.

The equator of $G(\mathbf{a},\mathbf{q})$ is the equatorial image of $\mathring{G}(\mathbf{a},\mathbf{q})$ within $G(\mathbf{a},\mathbf{q})$, that is the set of points $\{(x_1,\ldots,x_{d-1},\frac{\pi}{2}) \mid \mathbf{x} \in \mathring{G}(\mathbf{a},\mathbf{q})\}$.

Finally, every point of $\Pi\,\mathring{G}(\mathbf{a},\mathbf{q})$ is at spherical distance $\frac{\pi}{2}$ from both of the poles of $\mathbb{S}^d$ and therefore lies on the equator of $G(\mathbf{a},\mathbf{q})$. $\square$

Proof of Lemma 2.3.8.

It is easy to see that

$$\Pi\left(X \cap \mathring{G}(\mathbf{a},\mathbf{q})\right) \subseteq \Pi\,X \cap \Pi\,\mathring{G}(\mathbf{a},\mathbf{q}).$$

If $\mathbf{x} \in X \cap \mathring{G}(\mathbf{a},\mathbf{q})$ then $\mathbf{x} \in X$ and $\mathbf{x} \in \mathring{G}(\mathbf{a},\mathbf{q})$, so $\Pi\,\mathbf{x} \in \Pi\,X \cap \Pi\,\mathring{G}(\mathbf{a},\mathbf{q})$.

Now suppose that $\mathbf{y} \in \Pi\,X \cap \Pi\,\mathring{G}(\mathbf{a},\mathbf{q})$. Therefore there exist $\mathbf{x} \in X$, $\mathbf{g} \in \mathring{G}(\mathbf{a},\mathbf{q})$ such that

$$\mathbf{y} = \Pi\,\mathbf{x} = \Pi\,\mathbf{g}.$$

Therefore $\mathbf{x}$ and $\mathbf{g}$ lie on the same meridian. But $\oplus(\mathbf{g}) \in \Pi\,\mathring{G}(\mathbf{a},\mathbf{q})$ and $\mathbf{x} \in \oplus(\mathbf{x}) = \oplus(\mathbf{g})$. Therefore $\mathbf{x} \in X \cap \mathring{G}(\mathbf{a},\mathbf{q})$, and therefore

$$\mathbf{y} \in \Pi\left(X \cap \mathring{G}(\mathbf{a},\mathbf{q})\right).$$

$\square$

Proof of Lemma 2.3.9.

Let $C = \mathsf{S}(\mathbf{a},\Phi)$. Note that since $C \subset \mathring{\mathbb{S}}^2$, the spherical cap C does not contain either pole, and therefore every point of C has a meridian passing through it. Each meridian through a point of C also passes through $\partial(C)$ and therefore $\Pi\,\partial C = \Pi\,C$.

Since C does not contain either pole, we must have $\Phi < \frac{\pi}{2}$. Therefore every point of C has longitude between $\alpha_1 - \frac{\pi}{2}$ and $\alpha_1 + \frac{\pi}{2}$ (mod 2π).

We now note that two there are two meridians tangential to ∂C, to the west and east of $\mathbf{a}$. Denote by $\mathbf{w}$ the point of tangency with longitude less than $\mathbf{a}$, and denote by $\mathbf{e}$ the point of tangency with longitude more than $\mathbf{a}$ (mod 2π).

The corresponding meridians are $\oplus(\mathbf{w})$ and $\oplus(\mathbf{e})$. Since $\mathbf{a}$ is the centre of C, we can use reflection symmetry through the meridian $\oplus(\mathbf{a})$ to show that $\oplus(\mathbf{w})$ and $\oplus(\mathbf{e})$ make equal angles with $\oplus(\mathbf{a})$. Call the North pole $\mathbf{O}$ and define the angle $\phi := \angle\mathbf{wOa}$. Then $\angle\mathbf{aOe} = \phi$.

Thus the meridian $\oplus(\mathbf{w})$ meets the equator at the point $\Pi\,\mathbf{w}$ with longitude $\alpha_1 - \phi$ (mod 2π) and similarly the point $\Pi\,\mathbf{e}$ has longitude $\alpha_1 + \phi$ (mod 2π). Thus the equatorial image of C is

$$\Pi\,C = \odot\left([\alpha_1 - \phi, \alpha_1 + \phi] \pmod{2\pi}\right) = \mathsf{S}^1(\Pi\,\mathbf{a}, \phi).$$

The statements above can be verified by using stereographic projection based on the South pole of $\mathbb{S}^2$. to $\mathbb{R}^2$ [22, Section 5.2.4, pp. 223-227, Section 7.4.1, pp. 349–351], projecting the South pole to infinity and the North pole to the origin. We then use Euclidean geometry on the plane.

We now examine the angle ϕ. The North pole $\mathbf{O}$ and the points $\mathbf{a}$ and $\mathbf{e}$ form a spherical triangle, bounded by the meridians $\varphi(\mathbf{a})$ and $\varphi(\mathbf{e})$ and the geodesic $\mathbf{ae}$. By the well known sine formula for spherical triangles [22, Chapter 7, Theorem 8, p. 348] we have

$$\frac{\sin \mathsf{s}(\mathbf{O},\mathbf{e})}{\sin \angle \mathbf{eaO}} = \frac{\sin \mathsf{s}(\mathbf{e},\mathbf{a})}{\sin \angle \mathbf{aOe}} = \frac{\sin \mathsf{s}(\mathbf{O},\mathbf{a})}{\sin \angle \mathbf{Oea}}. \tag{2.12.1}$$

Since $\mathbf{a}$ is the centre of the cap C and since the meridian $\varphi(\mathbf{e})$ is tangential to ∂C at $\mathbf{e}$, $\varphi(\mathbf{e})$ meets the geodesic $\mathbf{ae}$ at a right angle. We therefore have $\angle \mathbf{Oea} = \frac{\pi}{2}$ and

$$\frac{\sin \Phi}{\sin \phi} = \frac{\sin \mathsf{s}(\mathbf{e},\mathbf{a})}{\sin \angle \mathbf{aOe}} = \sin \mathsf{s}(\mathbf{O},\mathbf{a}) = \sin \theta. \tag{2.12.2}$$

□

Proof of Lemma 2.3.10.

Note that since $\mathsf{S}(\mathbf{a},\Phi) \subset \mathring{\mathbb{S}}^d$, the spherical cap $\mathsf{S}(\mathbf{a},\Phi)$ does not contain either pole, and therefore every point of $\mathsf{S}(\mathbf{a},\Phi)$ has a meridian passing through it. Each meridian through a point of $\mathsf{S}(\mathbf{a},\Phi)$ also passes through $\partial\mathsf{S}(\mathbf{a},\Phi)$ and therefore $\Pi\,\partial\mathsf{S}(\mathbf{a},\Phi) = \Pi\,\mathsf{S}(\mathbf{a},\Phi)$.

Now consider the meridian $\varphi(\mathbf{a})$ and a point $\mathbf{q} \in \mathring{\mathbb{S}}^d$ which does not lie on the great circle defined by $\varphi(\mathbf{a})$. As per Lemma 2.3.7, these define the great $\mathbb{S}^2$, $G(\mathbf{a},\mathbf{q})$.

We now consider the intersection

$$\mathsf{S}^{\mathbf{q}}(\mathbf{a},\Phi) := \mathsf{S}(\mathbf{a},\Phi) \cap G(\mathbf{a},\mathbf{q}). \tag{2.12.3}$$

We see that $\mathsf{S}^{\mathbf{q}}(\mathbf{a},\Phi)$ is the spherical cap in $G(\mathbf{a},\mathbf{q})$ with centre $\mathbf{a}$ and spherical radius Φ, since $\mathsf{S}^{\mathbf{q}}(\mathbf{a},\Phi)$ is the intersection of the $\mathbb{S}^d$ spherical cap with centre $\mathbf{a}$ and spherical radius Φ with $G(\mathbf{a},\mathbf{q})$ which is a great $\mathbb{S}^2$ through $\mathbf{a}$.

We also have

$$\partial\mathsf{S}^{\mathbf{q}}(\mathbf{a},\Phi) = \partial\mathsf{S}(\mathbf{a},\Phi) \cap G(\mathbf{a},\mathbf{q}). \tag{2.12.4}$$

□

Proof of Lemma 2.3.13.

For a RISC region R which does not intersect either the North or South poles of $\mathbb{S}^d$, each side facet is the disjoint union of a set of meridian arcs. Each of the meridian arcs which make up a side facet is a geodesic which joins a point of the top facet to the corresponding point of the bottom facet, eg.

$$\mathcal{F}_{2,\downarrow}\,\mathsf{R} = \bigcup_{\alpha \in [\tau_1, \upsilon_1] \times \{\upsilon_2\} \times \ldots \times [\tau_{d-1}, \upsilon_{d-1}]} \odot \big(\{\alpha\} \times [\tau_d, \upsilon_d]\big).$$

Each of the meridian arcs which make up the side facets of the region R is part of a meridian which meets the equator at the boundary $\partial \Pi\,\mathsf{R}$ of $\Pi\,\mathsf{R}$. In fact $\partial \Pi\,\mathsf{R}$ is the disjoint union of all such intersections between these meridians and the equator, since every point of $\partial \Pi\,\mathsf{R}$ corresponds to a meridian through at least one of the side facets of R.

This is straightforward to prove if all of the facets of R are boundary facets. Now note that a degenerate facet of R corresponds to a degenerate facet of $\Pi\,\mathsf{R}$, and a pair of coincident facets of R corresponds to a pair of coincident facets of $\Pi\,\mathsf{R}$.

□

The area of a spherical cap

Proof of Lemma 2.3.14.

From (2.3.37) we have

$$\mathcal{V}_{E,d}(R) = \omega_{d-1} \int_0^{\Upsilon^{-1}(R)} \sin^{d-1}\theta \; d\theta.$$

We use the change of variables $r = \Upsilon(\theta) = 2 \sin \frac{\theta}{2}$, giving

$$\frac{d\,r}{d\,\theta} = \cos\frac{\theta}{2} = \sqrt{1 - \sin^2\left(\frac{\theta}{2}\right)} = \sqrt{1 - \frac{r^2}{4}},$$

since $\theta \in [0, \pi]$. We therefore obtain

$$d\theta = \frac{dr}{\sqrt{1 - \frac{r^2}{4}}}.$$

Also

$$\sin\theta = 2\sin\frac{\theta}{2}\ \cos\frac{\theta}{2} = r\sqrt{1 - \frac{r^2}{4}}.$$

We can therefore express the area element in terms of r as

$$\omega_{d-1}(\sin\theta)^{d-1}\ d\theta = \omega_{d-1} r^{d-1}\left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1}\ dr, \tag{2.12.5}$$

and therefore

$$\mathcal{V}_{E,d}(R) = \omega_{d-1}\int_0^R r^{d-1}\left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1}\ dr.$$

$\square$

Proof of Lemma 2.3.15.

From (2.3.35) we know that

$$\frac{\mathcal{V}_d(\theta)}{\mathcal{V}_d(\pi)} = \frac{\int_0^\theta \sin^{d-1}\xi\ d\xi}{\int_0^\pi \sin^{d-1}\xi\ d\xi}.$$

Now substitute $u = \sin^2(\xi/2)$. Then, since $\theta \in [0, \pi]$, by a well-known half angle formula we have $\sin\xi = 2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}$ and we also have $du = u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}d\xi$, so

$$\frac{\mathcal{V}_d(\theta)}{\mathcal{V}_d(\pi)} = \frac{\int_0^{\sin^2(\frac{\theta}{2})} u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\ du}{\int_0^1 u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\ du} = \frac{B\left(\sin^2\frac{\theta}{2}; \frac{d}{2}, \frac{d}{2}\right)}{B\left(\frac{d}{2}, \frac{d}{2}\right)}.$$

$\square$

The properties of the function $\mathcal{V}$ as given by Lemma 2.3.16 are well known, but we provide a proof here for completeness.

Proof of Lemma 2.3.16.

In order,

1. $\mathcal{V}$ is smooth since it is an integral of a smooth function over an interval.
2. $\mathcal{V}$ is monotonic increasing in $(0, \pi)$, since, using (2.3.41) $D\mathcal{V}(\theta) = \omega_{d-1} \sin^{d-1} \theta$, which is positive for $\theta \in (0, \pi)$.
3. $\mathcal{V}(0) = 0$ trivially. Property 3 then implies that $\mathcal{V}(\theta) > 0$ for $\theta > 0$.
4. We have $D^2\mathcal{V}(\theta) = (d-1)\omega_{d-1} \sin^{d-2} \theta \cos \theta$, which is positive in $(0, \pi/2)$ since both $\sin$ and $\cos$ are positive there. Therefore $D\mathcal{V}$ is monotonic increasing in $(0, \pi/2)$.
5. From properties 3 and 5, we therefore have, for $0 < h < \pi/2$,
$$0 < hD\mathcal{V}(\theta) \leqslant \int_{\theta}^{\theta+h} D\mathcal{V}(\xi)\, d\xi = \mathcal{V}(\theta + h) - \mathcal{V}(\theta) \leqslant hD\mathcal{V}(\theta + h),$$
for $\theta \in [0, \pi/2 - h]$.
6. Property 6 implies in particular that
$$\mathcal{V}(h) \leqslant hD\mathcal{V}(h),$$
for $h \in [0, \pi/2]$, since $\mathcal{V}(0) = 0$. We then have, for $h \leqslant \theta \leqslant \pi/2 - h$,
$$\mathcal{V}(\theta + h) - \mathcal{V}(\theta) \geqslant hD\mathcal{V}(\theta) \geqslant hD\mathcal{V}(h) \geqslant \mathcal{V}(h),$$
and for $\theta \leqslant h \leqslant \pi/2 - \theta$,
$$\mathcal{V}(\theta + h) - \mathcal{V}(h) \geqslant \theta D\mathcal{V}(h) \geqslant \theta D\mathcal{V}(\theta) \geqslant \mathcal{V}(\theta),$$
so for $0 \leqslant \theta + h \leqslant \pi/2$,
$$\mathcal{V}(\theta) + \mathcal{V}(h) \leqslant \mathcal{V}(\theta + h).$$
7. Using Property 2, we have $D\mathcal{V}(\theta) = D\mathcal{V}(\pi - \theta)$ since $\sin(\theta) = \sin(\pi - \theta)$.

8. Using property 8, we have

$$\mathcal{V}(\theta) = \omega_{d-1} \int_0^\theta \sin^{d-1}\xi \, d\xi = \omega_{d-1} \int_0^\theta \sin^{d-1}(\pi - \xi) \, d\xi$$
$$= \omega_{d-1} \int_{\pi-\theta}^{\pi} \sin^{d-1}\xi \, d\xi.$$

Therefore,

$$\mathcal{V}(\theta) + \mathcal{V}(\pi - \theta) = \omega_{d-1} \int_0^\pi \sin^{d-1}\xi \, d\xi = \omega_d.$$

$\square$

Proof of Lemma 2.3.17.

Using (2.3.44) with $\vartheta = \Theta_d(v)$, we have $\mathcal{V}_d(\pi - \Theta_d(v)) = \omega_d - \mathcal{V}_d(\Theta_d(v)) = \omega_d - v$, so

$$\pi - \Theta_d(v) = \Theta_d(\omega_d - v).$$

$\square$

Estimates

Proof of Lemma 2.3.18.

We see immediately that $\mathcal{V}(0) = 0$. For $\theta \in (0, \pi/2)$, since $\cos\xi > \cos\theta$ for $\xi \in [0, \theta)$, this gives us

$$\cos\theta \ \mathcal{V}(\theta) = \omega_{d-1} \int_0^\theta \cos\theta \ \sin^{d-1}\xi \ d\xi < \omega_{d-1} \int_0^\theta \cos\xi \ \sin^{d-1}\xi \ d\xi = \omega_{d-1} \frac{\sin^d\theta}{d}.$$

$\square$

Proof of Lemma 2.3.20.

For $d \geqslant 2$ we use (2.2.14) to obtain

$$\frac{\omega_d \ d}{\omega_{d-1}} = \frac{2 \ \pi^{\frac{d+1}{2}} \ d}{\Gamma\left(\frac{d+1}{2}\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{2 \ \pi^{\frac{d}{2}}} = d\sqrt{\pi} \ \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} > d\sqrt{\pi} \ \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)} = \sqrt{2\pi \ d}.$$

□

Proof of Lemma 2.3.21.

As a result of Lemma 2.3.14 and (2.3.53), the normalized area $\overset{*}{\mathcal{V}}_{E,d}(R)$ is given by

$$\overset{*}{\mathcal{V}}_{E,d}(R) = \frac{\omega_{d-1}}{\omega_d} \int_0^R r^{d-1}\left(1-\frac{r^2}{4}\right)^{\frac{d}{2}-1} dr.$$

For $R \in (0,T]$, $T \in (0,2]$, we have

$$1-\frac{r^2}{4} \in \left[1-\frac{T^2}{4},1\right).$$

The result follows immediately. □

Proof of Lemma 2.3.22.

Assume $d \geqslant 2$. By (2.3.35),

$$\frac{\omega_{d-1}}{d} = \frac{2\ \pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right) d} = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \in \left[\left(\frac{2\pi}{d}\right)^{\frac{d}{2}}, \pi^{\frac{d}{2}}\right],$$

where the upper bound results from (2.2.15). The result (2.3.56) follows immediately. □

Proof of Lemma 2.3.23.

Assume $d \geqslant 2$. By (2.3.56) we have

$$\left(1+\frac{1}{\sqrt{8\pi d}}\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d \geqslant \left(1+\frac{1}{2d}\right)^d > 1+\frac{1}{2}.$$

We also have

$$\omega_{d-1}\left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}}+1\right)^{d-1} = \frac{\left(1+\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d}{1+\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}}\ d\ \left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}} \geqslant \frac{1+\sqrt{2\pi d}}{1+\sqrt{\pi}}\ \sqrt{2\pi d} > 1.$$

□

2.12.3 Separation and packing

Proof of Lemma 2.7.8.

Define

$$\rho := \frac{\Upsilon^{-1}(\Delta)}{2} = \sin^{-1}\left(\frac{\Delta}{2}\right) \tag{2.12.6}$$

We therefore have $\rho \leqslant \operatorname{prad} X$, the packing radius of X. This implies that we can place each codepoint $\mathbf{y}$ of X in a spherical cap $\mathsf{S}(\mathbf{y}, \rho)$ with no two caps overlapping. This places an upper bound on g, of the form

$$\begin{aligned} g(R) &= \left|X \cap \mathsf{S}(\mathbf{x}, \Upsilon^{-1}(R))\right| \\ &\leqslant \frac{\overset{*}{\sigma}\big(\mathsf{S}(\mathbf{x}, \Upsilon^{-1}(R) + \rho)\big)}{\overset{*}{\sigma}\big(\mathsf{S}(\mathbf{x}, \rho)\big)} = \frac{\overset{*}{\mathcal{V}}_E\big(\,\Upsilon(\Upsilon^{-1}(R) + \rho)\big)}{\overset{*}{\mathcal{V}}_E\big(\,\Upsilon(\rho)\big)} \end{aligned}$$

where we have used (2.3.37) and (2.3.53) at the last step.

Since $\rho \in (0, \frac{\pi}{2}]$ we have

$$\Upsilon(\rho) = 2\ \sin\frac{\rho}{2} > \sin\rho = \frac{\Upsilon(2\rho)}{2} = \frac{\Delta}{2},$$

and since (2.3.5) gives us

$$\Upsilon^{-1}(R) + \rho < \Upsilon^{-1}(R) + 2\rho = \Upsilon^{-1}(R) + \Upsilon^{-1}(\Delta) < \Upsilon^{-1}(R + \Delta),$$

we see that

$$\overset{*}{\mathcal{V}}_E\big(\,\Upsilon(\rho)\big) > \overset{*}{\mathcal{V}}_E\left(\frac{\Delta}{2}\right) \quad \text{and} \quad \overset{*}{\mathcal{V}}_E\big(\,\Upsilon(\Upsilon^{-1}(R) + \rho)\big) < \overset{*}{\mathcal{V}}_E(R + \Delta).$$

From (2.12.7) we therefore have

$$g(R) \leqslant \frac{\overset{*}{\mathcal{V}}_E(R + \Delta)}{\overset{*}{\mathcal{V}}_E\left(\frac{\Delta}{2}\right)}.$$

Since $\Delta \leqslant 2$, (2.3.54) and (2.3.55) give us

$$g(R) \leqslant \frac{\mathsf{C}_{H,d}\ (R+\Delta)^d}{\mathsf{C}_{L,d}(1)\left(\frac{\Delta}{2}\right)^d} = 2^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} \left(\frac{R+\Delta}{\Delta}\right)^d. \tag{2.12.7}$$

This establishes (2.7.13).

To obtain (2.7.14) we note that for $R \leqslant \Delta$ we must have $g(R) = 1$ since the cap contains $\mathbf{x}$ itself. For $R > \Delta$ we have $2R > R + \Delta$ and so

$$\left(\frac{R+\Delta}{\Delta}\right)^d < 2^d \left(\frac{R}{\Delta}\right)^d.$$

□

2.12.4 Energy, weak-star convergence

Proof of Lemma 2.11.4.

Given N_0, define N_0+1 continuous non-negative functions f_1 to f_{N_0+1} with disjoint support, such that for each $j \in \{1, \ldots, N_0+1\}$,

$$\int_{\mathbb{S}^d} f_j(\mathbf{y}) d\overset{*}{\sigma}(\mathbf{y}) = 1.$$

Now define $\epsilon < 1$. Since $\mathcal{X}$ is weak-star convergent, this means that for each $j \in \{1, \ldots, N_0+1\}$, there is an L_j such that for all $\ell \geqslant L_j$ we have

$$\left| \int_{\mathbb{S}^d} f_j(\mathbf{y}) d\overset{*}{\sigma}_{X_\ell}(\mathbf{y}) - 1 \right| < \epsilon. \tag{2.12.8}$$

The inequality (2.12.8) implies that at least one point of X_ℓ is contained in the support of f_j. Now take $L_0 = \max(L_1, \ldots, L_{N+1})$. For $\ell \geqslant L_0$ we therefore must have at least one point of X_ℓ in each of the $N_0 + 1$ disjoint sets which are the supports of $\{f_1, \ldots, f_{N_0+1}\}$. □

Proof of Lemma 2.11.6.

This proof uses [122, Theorem 4.1, p. 665] with measures on $\mathbb{S}^d$. Given a sequence $\mathcal{X}$ of $\mathbb{S}^d$ codes, consider the corresponding sequence of normalized counting measures, $(\mathring{\sigma}_{X_\ell}, \ell \in \mathbb{N})$ and the normalized Lebesgue measure $\mathring{\sigma}$ on $\mathbb{S}^d$ as measures on $\mathbb{R}^{d+1}$.

Let L be the space of real linear functions on $\mathbb{R}^{d+1}$, restricted to $\mathbb{S}^d$. For $p \in L$ and $v \in \mathbb{R}$ the set $p^{-1}(v)$ is the preimage of v in $\mathbb{S}^d$. For the constant functions, this is either empty or is the whole of $\mathbb{S}^d$. The non-constant linear functions are of the form $p(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b$, with $\mathbf{a}$ non-zero. In $\mathbb{R}^{d+1}$ functions of this form have a preimage of the form $\{\mathbf{x} \in \mathbb{R}^{d+1} \mid \mathbf{a} \cdot \mathbf{x} = v - b\}$, which is a hyperplane orthogonal to $\mathbf{a}$, at distance $\frac{v-b}{\|a\|}$ from the origin. When p is restricted to $\mathbb{S}^d$ the corresponding preimage is the intersection of the hyperplane with $\mathbb{S}^d$, in other words, either the empty set, a single point or a small sphere. As v varies continuously, the distance from the origin of the corresponding hyperplane also varies continuously, as do the small spheres and their normalized Lebesgue measures. We have shown that $\mathring{\sigma} \circ p$ is continuous for each $p \in A$.

Following [122, 4. p. 665], let $\mathcal{H}_1$ be the class of half spaces of $\mathbb{R}^{d+1}$, that is, sets of the form $\{\mathbf{x} \in \mathbb{R}^{d+1} \mid p(\mathbf{x}) < v\}$ for some $p \in L$ and some $v \in \mathbb{R}$. If we have $A \in \mathcal{H}_1$ then $A \cap \mathbb{S}^d$ is the empty set, a point or a spherical cap, $\mathring{\sigma}_{X_\ell}(A) = \mathring{\sigma}_{X_\ell}(A \cap \mathbb{S}^d)$ and $\mathring{\sigma}(A) = \mathring{\sigma}(A \cap \mathbb{S}^d)$. Therefore

$$\sup_{A \in \mathcal{H}_1} \|\mathring{\sigma}_{X_\ell}(A) - \mathring{\sigma}(A)\| = \operatorname{disc}(X_\ell).$$

If a real function is continuous on $\mathbb{R}^{d+1}$ it is continuous on $\mathbb{S}^d$, and therefore $\mathring{\sigma}_{X_\ell}$ converges weakly to $\mathring{\sigma}$ on $\mathbb{R}^{d+1}$ if and only if it converges weakly to $\mathring{\sigma}$ on $\mathbb{S}^d$. Thus by [122, Theorem 4.1, p. 665], if a sequence $\mathcal{X}$ of $\mathbb{S}^d$ codes is weak-star convergent then the corresponding sequence of spherical cap discrepancies converges to zero.

One way to prove the converse is by first referring to [85, Theorem 3.3, p. 113], which gives an upper bound on the quadrature error of spherical harmonic polynomials in terms of the spherical cap discrepancy, and then using the Stone-Weierstrass theorem.

In more detail, for a $\mathbb{S}^d$ code X where $|X| = \mathcal{N}$, [85, (8), p. 109] defines a discrepancy based on spherical harmonics by

$$\operatorname{disc}^S(X) := \sup_{\mu \geqslant 1} \frac{1}{\mu^d} \sup_{j=1}^{\overset{*}{\mathcal{D}}(d,\mu)} \left| \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} Y_\mu^j(\mathbf{x}_k) \right|, \tag{2.12.9}$$

where Y_μ^j is the spherical harmonic of degree μ and order j, and where $\overset{*}{\mathcal{D}}(d,\mu)$ is the dimension of $\overset{*}{\mathbb{P}}_\mu(\mathbb{S}^d)$, the space of $\mathbb{S}^d$ polynomials of homogeneous degree μ, as per (2.6.4). In [85, Theorem 3.3, p. 113] it is shown that for any $\mathbb{S}^d$ code X where $|X| = \mathcal{N}$, the inequality

$$\operatorname{disc}^S(X) \leqslant \frac{2d(d+1)}{\omega_{d-1}\ \pi}\ \operatorname{disc} X \tag{2.12.10}$$

holds. We then take any polynomial $p \in \mathbb{P}_t(\mathbb{S}^d)$ and expand it in spherical harmonics to give

$$p := \sum_{\mu=0}^{t} \sum_{j=1}^{\overset{*}{\mathcal{D}}(d,\mu)} c_{(\mu,j)} Y_\mu^j.$$

The triangle inequality, (2.12.9) and (2.12.10) then yield

$$\begin{aligned}
\left| \int_{\mathbb{S}^d} p(\mathbf{x}) d\overset{*}{\sigma}(\mathbf{x}) - \int_{\mathbb{S}^d} p(\mathbf{x}) d\overset{*}{\sigma}_X(\mathbf{x}) \right| &\leqslant \sum_{\mu=0}^{t} \sum_{j=1}^{\overset{*}{\mathcal{D}}(d,\mu)} |c_{(\mu,j)}| \left| \int_{\mathbb{S}^d} Y_\mu^j(\mathbf{x}) d\overset{*}{\sigma}(\mathbf{x}) - \int_{\mathbb{S}^d} Y_\mu^j(\mathbf{x}) d\overset{*}{\sigma}_X(\mathbf{x}) \right| \\
&\leqslant \sum_{\mu=1}^{t} \sum_{j=1}^{\overset{*}{\mathcal{D}}(d,\mu)} |c_{(\mu,j)}|\, \mu^d\ \operatorname{disc}^S(X) \\
&\leqslant \sum_{\mu=1}^{t} \sum_{j=1}^{\overset{*}{\mathcal{D}}(d,\mu)} |c_{(\mu,j)}|\, \mu^d\ \frac{2d(d+1)}{\omega_{d-1}\ \pi}\ \operatorname{disc}(X) \\
&\to 0 \quad \text{as } \operatorname{disc}(X) \to 0.
\end{aligned}$$

We now use [93, Theorem 0.4, p. 7] which applies to vague convergence of measures. As per [93, p. 3], vague convergence is defined in terms of $C_c(\mathbb{S}^d)$, the space of real continuous functions on $\mathbb{S}^d$ with compact support. But since $\mathbb{S}^d$ is itself compact, $C_c(\mathbb{S}^d)$ coincides with $C(\mathbb{S}^d)$, the space of real continuous functions on $\mathbb{S}^d$. The definition of [93, (0.1.7), p. 7] as applied to $\mathbb{S}^d$ therefore coincides with Definition 2.11.1, and we can therefore apply [93, Theorem 0.4, p. 7] here. We define $C^+(\mathbb{S}^d)$ to be the non-negative real continuous functions on $\mathbb{S}^d$. As applied to our case, [93, Theorem 0.4, p. 7] then states that if the set $M \subset C^+(\mathbb{S}^d)$ is dense in $C^+(\mathbb{S}^d)$ and if

$$\int_{\mathbb{S}^d} f(\mathbf{x})\ d\nu_\ell(\mathbf{x}) \to \int_{\mathbb{S}^d} f(\mathbf{x})\ d\nu(\mathbf{x}),$$

for any function $f \in M$, then ν_ℓ weakly converges to ν.

Finally, we apply the Stone-Weierstrass theorem [90, Lemma 1.1, p. 173], which in our case states that the polynomials on $\mathbb{S}^d$ are dense in $C(\mathbb{S}^d)$. With a little effort, this can be used to show that the non-negative polynomials on $\mathbb{S}^d$ are dense in $C^+(\mathbb{S}^d)$. (See also [124, Theorem 5.8, p. 121] for a proof of Weierstrass' theorem which is sufficient for our purposes here.) $\square$

Proof of Lemma 2.11.8.

Fix $d > 1$ and use the abbreviation $\mathcal{J} := \mathcal{J}_d$. For any particular $\mathbf{x}, \mathbf{y}$, define $r := \|\mathbf{x} - \mathbf{y}\|$, $\theta := \Upsilon^{-1}(r)$, so that r is the Euclidean distance from $\mathbf{x}$ to $\mathbf{y}$, and θ is the spherical distance. The expression $\mathcal{J}(\mathbf{x}; R)\, u$ is an integral over the spherical cap $\mathsf{S}_E(\mathbf{x}, R)$. From the proof of Lemma 2.3.14 and from (2.12.5) we see that the relevant area element for the integral $\mathcal{J}(\mathbf{x}; R)\, u$ is

$$\frac{\omega_{d-1}}{\omega_d} (\sin\theta)^{d-1}\ d\theta = \frac{\omega_{d-1}}{\omega_d} r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1}\ dr.$$

We therefore have

$$\mathcal{J}(\mathbf{x}; R)\, u = \frac{\omega_{d-1}}{\omega_d} \int_0^R u(r)\ r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1}\ dr = \mathcal{J}(R)\, u$$

independent of $\mathbf{x}$. $\square$

Chapter 3

Equal area partitions

"And so we two shall all love's lemmas prove,

And in our bound partition never part."

– Lem [95, Love and Tensor Algebra, pp. 52–53].

3.1 Introduction

This chapter describes a partition of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ which is here called the recursive zonal equal area (EQ) partition. Parts of this chapter appear in [99].

The partition $\mathrm{EQ}(d, \mathcal{N})$ is a partition of the unit sphere $\mathbb{S}^d$ into $\mathcal{N}$ regions of equal area and small diameter. It is defined via the algorithm given in Section 3.2.

Figure 3.1 shows an example of the partition $\mathrm{EQ}(2, 33)$, the recursive zonal equal area partition of $\mathbb{S}^2$ into 33 regions. A movie showing the build-up of an example of the partition $\mathrm{EQ}(3, 99)$ is available via the author's web site at UNSW [100].

Definition 3.1.1. *The set of recursive zonal equal area partitions of $\mathbb{S}^d$ is defined as*

$$\mathrm{EQ}(d) := \{\mathrm{EQ}(d, \mathcal{N}) \mid \mathcal{N} \in \mathbb{N}_+\}. \tag{3.1.1}$$

where $\mathrm{EQ}(d, \mathcal{N})$ denotes the recursive zonal equal area partition of the unit sphere $\mathbb{S}^d$ into $\mathcal{N}$ regions, which is defined via the algorithm given in Section 3.2.

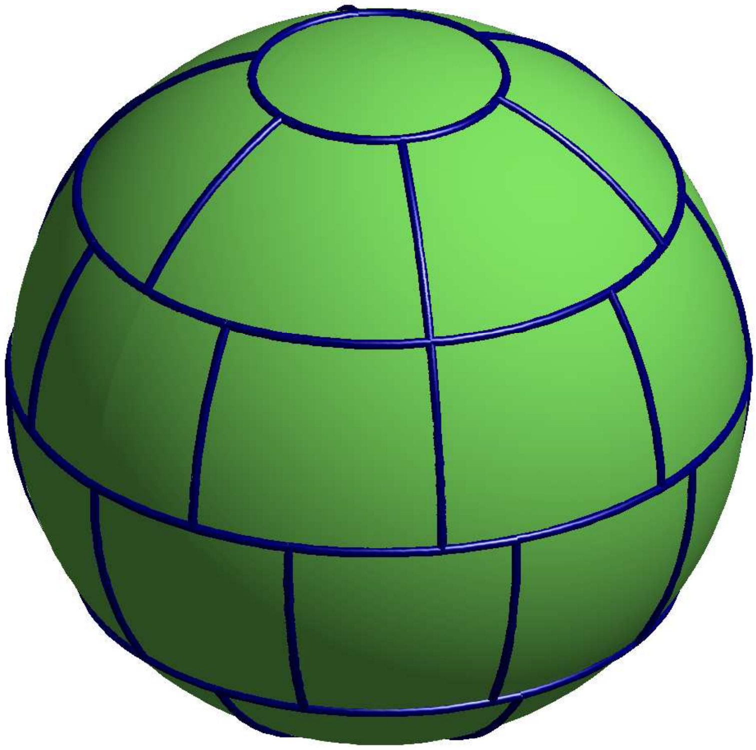

Figure 3.1: Partition $\mathrm{EQ}(2, 33)$

Later in this section we prove that the partition defined via the algorithm given in Section 3.2 has the following properties.

Theorem 3.1.2. *For $d \geqslant 1$ and $\mathcal{N} \geqslant 1$, the partition $\mathrm{EQ}(d,\mathcal{N})$ is an equal area partition of $\mathbb{S}^d$.*

Theorem 3.1.3. *For $d \geqslant 1$, $\mathrm{EQ}(d)$ is diameter-bounded in the sense of Definition 2.4.3.*

The proof of Theorem 3.1.2 is straightforward, following immediately from the construction of Section 3.2. A sketch of the proof of Theorem 3.1.3 is given in Section 3.4, and the full proof is given in Section 3.6.

The construction for the recursive zonal equal area partition is based on Zhou's construction for $\mathbb{S}^2$ [167], as modified by Saff [133], and on Sloan's notes on the partition of $\mathbb{S}^3$ [141].

The existence of partitions of $\mathbb{S}^d$ into regions of equal area and small diameter is well known and has been used in a number of ways. Alexander [3, Lemma 2.4

p. 447] uses such a partition of $\mathbb{S}^2$ to derive a lower bound for the maximum sum of distances between points. The paper also suggests a construction for $\mathbb{S}^2$ [3, p. 447], which differs from Zhou's construction. For $6m^2$ regions, Alexander begins with a spherical cube which divides $\mathbb{S}^2$ into 6 regions, then divides each face into m slices by using a pencil of $m-1$ great circles with positions adjusted so that each slice has the same area. Finally, each slice is divided into m regions of equal area by another pencil of $m-1$ great circles, which may differ for each slice. Alexander then asserts that the diameters are the right magnitude and omits a proof. This construction has an obvious generalization for $\mathbb{S}^d$ with $2(d+1)m^d$ regions. Start with the appropriate spherical hypercube, then divide each face into m equal pieces, and so on. It is not clear that this partition of $\mathbb{S}^d$ is diameter-bounded in the sense of Definition 2.4.3.

The existence of a diameter bounded set of equal area partitions of $\mathbb{S}^d$ is used by Stolarsky [147], Beck and Chen [6] and Bourgain and Lindenstrauss [20], but no construction is given.

Stolarsky [147, p. 581] asserts the existence of such a set, saying simply,

> "Now clearly one can choose the A_i so that their Euclidean diameters are $\gg\ll \mathcal{N}^{-\frac{1}{m-1}}$ for $1 \leqslant i \leqslant N$."

Here Stolarsky is discussing a partition of $\mathbb{S}^{m-1}$ into N regions labelled A_i. Stolarsky's notation $\gg\ll$ is equivalent to order notation, and his assertion can be restated as:

> There are constants $c, C > 0$ such that for any $N > 0$ one can choose the regions A_i so that their Euclidean diameters are bounded by $cN^{-\frac{1}{m-1}} \leqslant \operatorname{diam} A_i \leqslant CN^{-\frac{1}{m-1}}$ for $1 \leqslant i \leqslant N$.

The paper then uses this assertion to prove a theorem which relates the sum of distances between N points on $\mathbb{S}^{m-1}$ to a discrepancy which is defined in the paper.

Beck and Chen [6, pp. 237–238] essentially cites Stolarsky's result, asserting that

"One can easily find a partition

$$\mathbb{S}^d = \bigcup_{\ell=1}^{\mathcal{N}} \mathsf{R}_\ell$$

such that for $1 \leqslant \ell \leqslant \mathcal{N}$, $\sigma(\mathsf{R}_\ell) = \frac{\sigma(\mathbb{S}^d)}{\mathcal{N}}$ and $\operatorname{diam} \mathsf{R}_\ell \ll \mathcal{N}^{-\frac{1}{d}}$, where $\operatorname{diam} \mathsf{R}_\ell$ is the diameter of R_ℓ."

[With notation adjusted to match this thesis.]

Bourgain and Lindenstrauss [20, p. 26] cite Beck and Chen [6] and use a diameter-bounded equal area partition of $\mathbb{S}^{n-1}$ to prove their Theorem 1 on the approximation of zonoids by zonotopes.

Stolarsky's assertion can be proved using the method used by Feige and Schechtman [54] to prove the following lemma.

Lemma 3.1.4. *(Feige and Schechtman [54, Lemma 21, pp. 430–431])*

For each $0 < \gamma < \frac{\pi}{2}$ the sphere $\mathbb{S}^{d-1}$ can be partitioned into $N = \left(\frac{\mathrm{O}(1)}{\gamma}\right)^d$ regions of equal area, each of diameter at most γ.

Feige and Schechtman's proof is not fully constructive. The construction assumes the existence of an algorithm which creates a packing on the unit sphere having the maximum number of equal spherical caps of given spherical radius [164, p. 1091] [165, Lemma 1, p. 2112]. This assumption is not necessary for the proof, and a fully constructive proof is therefore possible. This is given here as the proof of Lemma 3.8.1 below.

Wagner [159, p. 112] implies that a diameter-bounded sequence of equal area partitions of $\mathbb{S}^d$ can be constructed where each region is a rectangular polytope in spherical polar coordinates. For $\mathbb{S}^2$, this is the same form of partition as [167] and [133], and for $\mathbb{S}^d$, this is the same form as given in this thesis.

Rakhmanov, Saff and Zhou [120], Zhou [167, 168] and Kuijlaars and Saff [134, 88] use the partition of $\mathbb{S}^2$ given by Zhou's construction to obtain bounds on the extremal energy of point sets.

Other constructions for equal area partitions of $\mathbb{S}^2$ have been used in the geosciences [84, 145] and astronomy [152, 37, 62], but these constructions do not have a proven bound on the diameter of regions. In particular, the regions of the "igloo" partitions of [37] have the same form as [167] and [133]. The paper [37] also discusses nesting schemes for "igloo" partitions.

The paper [96], on the problem of partitioning $\mathbb{S}^d$ into spherical quadrilaterals of equal area, describes in great detail a construction very similar to Alexander's construction [3], but does not reference [3]. It instead says "We have found no references on this problem in the literature".

The remainder of this chapter is organized as follows. Section 3.2 describes the partition algorithm. Section 3.3 presents an analysis of the regions of a partition, including a number of lemmas used to prove the main theorems. Section 3.4 gives a sketch of the proof of Theorem 3.1.3. Section 3.5 describes a continuous model of the partition algorithm. Section 3.6 presents the proof of the main theorems. Section 3.7 proves a per-region bound on diameters. Section 3.8 presents a proof of Stolarsky's assertion [147, p. 581] on the existence of equal area partitions of the sphere with small diameter. Section 3.9 describes the Matlab implementation of the EQ partition algorithm. Section 3.10 presents numerical results. Section 3.11 contains detailed proofs of lemmas. Section 3.12 contains estimates of the values of some constants resulting from Theorem 3.1.3 and its proof.

3.2 The recursive zonal equal area partition

This section describes the recursive zonal equal area partition and recursive zonal equal area partition algorithm in some detail.

3.2.1 The recursive zonal equal area partition algorithm in outline

The recursive zonal equal area partition algorithm is recursive in dimension d. The pseudocode description for the algorithm for $\mathrm{EQ}(d,\mathcal{N})$ is given by Figure 3.2.

if $\mathcal{N} = 1$ **then**
    There is a single region which is the whole sphere;
else if $d = 1$ **then**
    Divide the circle into $\mathcal{N}$ equal segments;
else
    Divide the sphere into zones, each the same area as an integer number of regions:
        1. Determine the colatitudes of polar caps,
        2. Determine an ideal collar angle,
        3. Determine an ideal number of collars,
        4. Determine the actual number of collars,
        5. Create a list of the ideal number of regions in each collar,
        6. Create a list of the actual number of regions in each collar,
        7. Create a list of colatitudes of each zone;
    Partition each spherical collar into regions of equal area,
        using the recursive zonal equal area partition algorithm for dimension $d-1$;
endif.

Figure 3.2: The recursive zonal equal area Partition algorithm

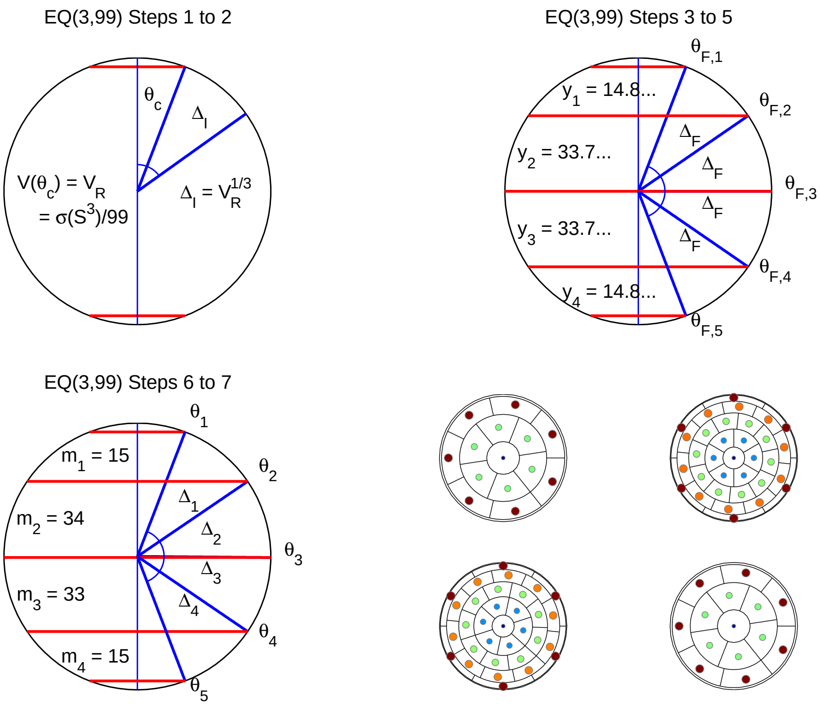

Figure 3.3: Partition algorithm for $\mathrm{EQ}(3, 99)$

Figure 3.3 is an illustration of the algorithm for $\mathrm{EQ}(3, 99)$, with step numbers corresponding to the step numbers in the pseudocode. We now describe key steps of the algorithm in more detail.

3.2.2 Dividing the sphere into zones

This is the key part of the algorithm, and is split into a number of steps. Each step is described in more detail below. For brevity, we assume $d > 1$ and $\mathcal{N} > 1$ and we omit mentioning dependence on the variables d and $\mathcal{N}$, where this can be done without confusion.

1. *Determining the colatitudes of polar caps.*

   Each polar cap is a spherical cap with the same area as that required for a region. For an $\mathcal{N}$ region partition of $\mathbb{S}^d$, the required area of a region R is

   $$\mathcal{V}_R := \frac{\omega_d}{\mathcal{N}}, \tag{3.2.1}$$

   where ω_d is the area of $\mathbb{S}^d$, as per (2.3.35).

   The colatitude of the bottom of the North polar cap, ϑ_c is the spherical radius of a spherical cap of area $\mathcal{V}_R$. Therefore

   $$\vartheta_c := \Theta(\mathcal{V}_R), \tag{3.2.2}$$

   where the function Θ is defined by (2.3.45). The colatitude of top of the South polar cap is then $\pi - \vartheta_c$.

2. *Determining an ideal collar angle.*

   As a result of Lemma 2.3.2, spherical distance approaches Euclidean distance as the distance goes to zero. We now use the idea that to keep the diameter bounded we want the shape of each region to approach a d-dimensional Euclidean hypercube as $\mathcal{N}$ goes to infinity. That way, the diameter approaches the diagonal length of the hypercube. The collar angle, the spherical distance between the top and bottom of a collar in the partition, therefore should approach $\mathcal{V}_R^{\frac{1}{d}}$ as $\mathcal{N}$ approaches infinity.

   We therefore define the *ideal collar angle* to be

   $$\delta_I := \mathcal{V}_R^{\frac{1}{d}}. \tag{3.2.3}$$

3. *Determining an ideal number of collars.*

   Ideally, the sphere is to be partitioned into the North and South spherical caps, and a number of collars, all of which have angle δ_I. The ideal number of collars is therefore

$$\mathsf{n}_I := \frac{\pi - 2\vartheta_c}{\delta_I}. \tag{3.2.4}$$

4. *Determining the actual number of collars.*

   We use a rounding procedure to obtain an integer n close to the ideal number of collars.

   If $\mathcal{N} = 2$, then $\mathsf{n} := 0$. Otherwise

$$\mathsf{n} := \max\left(1, \operatorname{round}\left(\mathsf{n}_I\right)\right), \tag{3.2.5}$$

   where, as usual, for $x \geqslant 0$,

$$\operatorname{round}(x) := \lfloor x + 0.5 \rfloor, \tag{3.2.6}$$

   where $\lfloor\rfloor$ is the floor (greatest integer) function.

   The number of collars is then n.

5. *Creating a list of the ideal number of regions in each collar.*

   We number the zones southward from 1 for the North polar cap to $\mathsf{n}+2$ for the South polar cap, and number the collars so that collar i is zone $i+1$.

   We now assume $\mathcal{N} > 2$. The *"fitting"* collar angle is

$$\delta_F := \frac{\mathsf{n}_I}{\mathsf{n}}\delta_I = \frac{\pi - 2\vartheta_c}{\mathsf{n}}. \tag{3.2.7}$$

   We use δ_F to produce an increasing list of "fitting" colatitudes of caps, defined by

$$\vartheta_{F,i} := \vartheta_c + (i-1)\delta_F, \tag{3.2.8}$$

for $i \in \{1, \ldots, \mathsf{n}+1\}$.

The area of each corresponding "fitting" collar is given by successive colatitudes in this list. The ideal number of regions, y_i, in each collar $i \in \{1, \ldots, \mathsf{n}\}$ is then

$$\mathsf{y}_i := \frac{\mathcal{V}(\vartheta_{F,i+1}) - \mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R}. \tag{3.2.9}$$

6. *Creating a list of the actual number of regions in each collar.*

   We use a rounding procedure similar to that of Zhou Lemma 2.11 [167, pp. 16–17]. With n the number of collars as defined by (3.2.5), we define m_i, the required number of regions in collar $i \in \{1, \ldots, \mathsf{n}\}$ as follows.

   Define the sequences a and m by starting with $\mathsf{a}_0 := 0$, and for $i \in \{1, \ldots, \mathsf{n}\}$,

$$\mathsf{m}_i := \operatorname{round}(\mathsf{y}_i + \mathsf{a}_{i-1}), \quad \mathsf{a}_i := \sum_{j=1}^{i} (\mathsf{y}_j - \mathsf{m}_j). \tag{3.2.10}$$

7. *Creating a list of colatitudes of each zone.*

   We now define $\vartheta_0 := 0$, $\vartheta_{\mathsf{n}+2} := \pi$ and for $i \in \{1, \ldots, \mathsf{n}+1\}$, we define

$$\vartheta_i := \Theta\left(\left(1 + \sum_{j=1}^{i-1} \mathsf{m}_j\right)\mathcal{V}_R\right). \tag{3.2.11}$$

   For $i \in \{0, \ldots, \mathsf{n}+1\}$, we use Z as per (2.3.16) to define zone i to be $Z(\vartheta_i, \vartheta_{i+1})$.

   Finally, for $i \in \{1, \ldots, \mathsf{n}\}$, we define collar i to be zone i.

3.2.3 Partitioning a collar

We partition collar i of $\mathrm{EQ}(d, \mathcal{N})$ into m_i regions, each corresponding to a region of the partition $\mathrm{EQ}(d-1, \mathsf{m}_i)$. We assume that each region of $\mathrm{EQ}(d-1, \mathsf{m}_i)$ is rectilinear in spherical polar coordinates (RISC) as per Definition 2.3.11. If region $j \in \{1, \ldots, \mathsf{m}_i\}$ of $\mathrm{EQ}(d-1, \mathsf{m}_i)$ is $\mathcal{R}((\tau_1, \ldots, \tau_{d-1}), (\upsilon_1, \ldots, \upsilon_{d-1}))$, then we define the region R of collar i of $\mathrm{EQ}(d, \mathcal{N})$ corresponding to region j of $\mathrm{EQ}(d-1, \mathsf{m}_i)$ to be

$$\mathsf{R} := \mathcal{R}\left((\tau_1, \ldots, \tau_{d-1}, \vartheta_i), (\upsilon_1, \ldots, \upsilon_{d-1}, \vartheta_{i+1})\right). \tag{3.2.12}$$

Remarks. The partition $\mathrm{EQ}(d,\mathcal{N})$ is not fully specified by this algorithm. The algorithm instead specifies an equivalence class of partitions, unique up to rotations of the sectors of the partitions of $\mathbb{S}^1$. This means that the collars of $\mathrm{EQ}(2,\mathcal{N})$ are free to rotate without changing diameters of the regions and without changing the colatitudes of the collars. The regions remain rectilinear in spherical polar coordinates.

3.3 Analysis of the recursive zonal equal area partition

The proofs of Theorems 3.1.2 and 3.1.3 proceed by induction on d, matching the recursion of the recursive zonal equal area partition algorithm. This section presents the preliminary analysis of the recursive zonal equal area partition, including the lemmas needed by the proofs of Theorems 3.1.2 and 3.1.3. Section 3.11 contains proofs of the lemmas.

First, we characterize the regions of a recursive zonal equal area partition. Following this, we examine the case $d > 1$, $\mathcal{N} > 1$ in some detail. The cases $d = 1$ and $\mathcal{N} = 1$ are simpler and are included in the proof itself.

By induction on the construction given in Section 3.2, we see that the regions produced by the recursive zonal equal area partition algorithm are RISC regions as per Definition 2.3.11, and for $d > 1$ each region R of collar i is of the form (3.2.12). Each such region therefore has an equatorial image of the form

$$\begin{aligned} \Pi\,\mathsf{R} &= \odot\left([\tau_1, \upsilon_1] \times [\tau_2, \upsilon_2] \times \ldots \times [\tau_{d-1}, \upsilon_{d-1}]\right) \qquad (3.3.1) \\ &= \mathcal{R}((\tau_1, \ldots, \tau_{d-1}), (\upsilon_1, \ldots, \upsilon_{d-1})) \in \mathrm{EQ}(d-1, \mathsf{m}_i) \end{aligned}$$

as per Lemma 2.3.12 and Section 3.2.3 above.

We now consider the two polar caps. The following lemma on the diameter of the polar caps has an elementary proof, which is included in Section 3.11 for completeness.

Lemma 3.3.1. *For $d > 1$ and $\mathcal{N} > 1$, the diameter of each of the polar caps of the recursive zonal equal area partition $\mathrm{EQ}(d,\mathcal{N})$ is $2\sin\vartheta_c$, where ϑ_c is defined by (3.2.2).*

An analysis of the diameter of the polar caps is not needed for the proof of Theorem 3.1.3. This is a consequence of the isodiametric inequality for $\mathbb{S}^d$.

Theorem 3.3.2. *(Isodiametric inequality for $\mathbb{S}^d$)*

Any region $\mathsf{R} \subset \mathbb{S}^d$ of spherical diameter $\delta < \pi$ has area bounded by

$$\sigma(\mathsf{R}) \leqslant \mathcal{V}\left(\frac{\delta}{2}\right).$$

Equality holds only for spherical caps of spherical radius $\frac{\delta}{2}$.

> *Remarks.* This result is well known. See [16] for a proof of a generalized version of this inequality, based on the proof of [7].
>
> The corresponding result for Euclidean space – with no restriction on diameter – is also well known [53, Corollary 2.10.33 p. 197].

Corollary 3.3.3. *The polar caps are the regions of smallest diameter of* $\mathrm{EQ}(d, \mathcal{N})$.

Considering Corollary 3.3.3, the polar caps need not be taken into account when estimating the maximum diameter of regions of $\mathrm{EQ}(d, \mathcal{N})$ for $\mathcal{N} > 2$. We therefore turn our attention to the regions contained in collars.

The following lemma leads to a bound on the diameter of a region contained in a collar.

Lemma 3.3.4. *Given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{S}^d$ where*

$$\mathbf{a} := \odot(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, A), \quad \mathbf{b} := \odot(\beta_1, \beta_2, \ldots, \beta_{d-1}, B), \quad \mathbf{c} := \odot(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B), \tag{3.3.2}$$

with $\sin B \geqslant \sin A$, then the Euclidean $\mathbb{R}^{d+1}$ distance $\|\mathbf{a} - \mathbf{b}\|$ satisfies

$$\|\mathbf{a} - \mathbf{b}\| \leqslant \sqrt{\|\mathbf{a} - \mathbf{c}\|^2 + \|\mathbf{c} - \mathbf{b}\|^2}.$$

The following definitions are of use in examining the diameter of R in terms of $\|\mathbf{a} - \mathbf{c}\|$ and $\|\mathbf{c} - \mathbf{b}\|$. For region R contained in collar i of $\mathrm{EQ}(d, \mathcal{N})$,

- the spherical distance between the top and bottom parallels of region R is

$$\delta_i := \vartheta_{i+1} - \vartheta_i, \tag{3.3.3}$$

- the maximum Euclidean radius of collar i is

$$\mathsf{w}_i := \max_{\xi \in [\vartheta_i, \vartheta_{i+1}]} \sin \xi = \begin{cases} \sin \vartheta_{i+1} & \text{if} \quad \vartheta_{i+1} < \frac{\pi}{2}, \\ \sin \vartheta_i & \text{if} \quad \vartheta_i > \frac{\pi}{2}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.3.4}$$

We can now use Lemmas 2.3.2 and 3.3.4 to show that

Lemma 3.3.5. *For region* R *contained in collar* i *of* $\mathrm{EQ}(d, \mathcal{N})$ *we have*

$$\begin{aligned} \operatorname{diam} \mathsf{R} &\leqslant \sqrt{\Upsilon(\delta_i)^2 + \mathsf{w}_i^2 (\operatorname{diam} \Pi\, \mathsf{R})^2} \\ &\leqslant \sqrt{\delta_i^2 + \mathsf{w}_i^2 (\operatorname{diam} \Pi\, \mathsf{R})^2}, \end{aligned}$$

where δ_i *and* w_i *are given by (3.3.3) and (3.3.4) respectively.*

3.4 Sketch of the proof of Theorem 3.1.3

The proof of Theorem 3.1.3 proceeds by induction on the dimension d. The inductive step of the proof starts with the observation that if $d > 1$ and if the set $\mathrm{EQ}(d-1)$ has diameter bound κ, then for any region R of collar i of the partition $\mathrm{EQ}(d, \mathcal{N})$ we have

$$\operatorname{diam} \Pi\, \mathsf{R} \leqslant \kappa \mathsf{m}_i^{\frac{1}{1-d}},$$

and therefore from Lemma 3.3.5 we have

$$\operatorname{diam} \mathsf{R} \leqslant \sqrt{\delta_i^2 + \kappa^2 \mathsf{p}_i^2},$$

where the scaled $\mathbb{S}^{d-1}$ diameter bound p_i is

$$\mathsf{p}_i := \mathsf{w}_i \mathsf{m}_i^{\frac{1}{1-d}}. \tag{3.4.1}$$

As a consequence, if $d > 1$ and if $\mathrm{EQ}(d-1)$ has diameter bound κ, then for any region R of the partition $\mathrm{EQ}(d, \mathcal{N})$

$$\operatorname{diam} \mathsf{R} \leqslant \sqrt{(\max \delta)^2 + \kappa^2 (\max \mathsf{p})^2}, \tag{3.4.2}$$

$$\text{where} \quad \max \delta := \max_{i \in \{1,\ldots,\mathsf{n}\}} \delta_i, \quad \max \mathsf{p} := \max_{i \in \{1,\ldots,\mathsf{n}\}} \mathsf{p}_i,$$

and n is the number of collars in the partition $\mathrm{EQ}(d, \mathcal{N})$.

Thus to prove the theorem it suffices to show that $\max \delta$ and $\max \mathsf{p}$ are both of order $\mathcal{N}^{-\frac{1}{d}}$. Since the Euclidean diameter of a region of $\mathbb{S}^d$ is always bounded above by 2, we need only prove that there is an $N_0 \geqslant 1$ such that for $\mathcal{N} \geqslant N_0$ we have bounds of the right order. This is because for any $N_0 \geqslant 1$ and any $\mathcal{N} \in [1, N_0]$ we have

$$2\, N_0^{\frac{1}{d}}\, \mathcal{N}^{-\frac{1}{d}} \geqslant 2.$$

> *Remarks.* Lemma 3.3.5 in its current form is not strictly necessary for the proof of Theorem 3.1.3. A lemma using the triangle inequality would suffice. The main reason for using Lemma 3.3.5 is to improve the value of the constant K_d of Theorem 3.1.3.

The key strategy in estimating $\max \delta$ and $\max \mathsf{p}$ is to replace the integer variable i by a small number of real valued variables constrained to some feasible domain, replace δ and p with the equivalent functions of these real variables, and then to find and estimate continuous functions which dominate these equivalent functions.

To replace i, we first must model the rounding steps of the partition algorithm. We model the first rounding step by finding appropriate bounds for $\rho = \frac{\mathsf{n}_I}{\mathsf{n}} = \frac{\delta_F}{\delta_I}$, where δ_F is defined by (3.2.7).

The second rounding step takes the sequence y and produces the sequences m and a. To model this step, we first show that $\mathsf{a}_i \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. This allows us to define the analog functions $\mathcal{Y}, \mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}$ corresponding to $\mathsf{y}, \mathsf{m}, \delta, \mathsf{w}, \mathsf{p}$ respectively. These analog functions are defined on the real rounding variables τ and β and the colatitude variable ϑ, such that $\mathcal{Y}$ coincides with y, etc. when $\tau = -\mathsf{a}_{i-1}$, $\beta = \mathsf{a}_i$ and $\vartheta = \vartheta_{F,i}$, where $\vartheta_{F,i}$ is defined by (3.2.8).

We then define the feasible domain $\mathbb{D}$ such that the second rounding step always corresponds to a set of points in $\mathbb{D}$.

The final and longest part of the proof is to show that both Δ and P are asymptotically bounded of order $\mathcal{N}^{-\frac{1}{d}}$ over the whole of $\mathbb{D}$. In this final part, we need estimates for the area function $\mathcal{V}$ and the inverse function Θ. Crude but very simple estimates of these functions yield bounds for Δ and P of the correct order.

3.5 A continuous model of the partition algorithm

3.5.1 Rounding the number of collars

For the first rounding step, which produces n from n_I, we define

$$\rho := \frac{\mathsf{n}_I}{\mathsf{n}} \tag{3.5.1}$$

so that

$$\delta_F = \rho \delta_I \tag{3.5.2}$$

We recall from (3.2.6) that for $x > 0$,

$$\operatorname{round}(x) \in \left(x - \frac{1}{2}, x + \frac{1}{2}\right] \tag{3.5.3}$$

Therefore, using (3.2.5) and (3.5.3), for $\mathcal{N} > 2$, if $\mathsf{n}_I \geqslant \frac{1}{2}$ then

$$\mathsf{n} \in \left(\mathsf{n}_I - \frac{1}{2}, \mathsf{n}_I + \frac{1}{2}\right]. \tag{3.5.4}$$

We can now prove the following.

Lemma 3.5.1. *For* $\mathcal{N} > 2$*, if* $\mathsf{n}_I > \frac{1}{2}$ *then*

$$\rho \in \left[1 - \frac{1}{2\mathsf{n}_I + 1}, 1 + \frac{1}{2\mathsf{n}_I - 1}\right). \tag{3.5.5}$$

Using (3.2.4) and Lemma 3.5.1, we see that bounds for ρ are given by lower bounds for n_I. The crudest bound is given by $\mathsf{n}_I > \frac{1}{2}$, for then

$$\rho \in \left[1 - \frac{1}{2\mathsf{n}_I + 1}, 1 + \frac{1}{2\mathsf{n}_I - 1}\right) \subset \left(\frac{1}{2}, \infty\right). \tag{3.5.6}$$

We can re-express the bound $\mathsf{n}_I > \frac{1}{2}$ in terms of a lower bound on $\mathcal{N}$ by means of the function ν, where

$$\nu(x) := \left(\frac{x}{\omega_d}\right)^{\frac{1}{d}} \left(\pi - 2\Theta\left(\frac{\omega_d}{x}\right)\right). \tag{3.5.7}$$

Lemma 3.5.2. *The function* ν *defined by (3.5.7) has the following properties.*

1. $\nu(2) = 0$.
2. $\nu(\mathcal{N}) = \mathsf{n}_I$.
3. $\nu(x)$ *is monotonically increasing in* x *for* $x \geqslant 2$.

As a consequence of Lemma 3.5.2, it is possible to define the inverse function $\mathcal{N}_0$ where

$$\mathcal{N}_0(y) := \nu^{-1}(y), \tag{3.5.8}$$

for $y \geqslant 0$. We then have $\mathcal{N}_0(\nu(x)) = x$ and $\nu(\mathcal{N}_0(y)) = y$ for $x \geqslant 2$ and $y \geqslant 0$, and by the inverse function theorem, $\mathcal{N}_0(y)$ is monotonic increasing in y for $y \geqslant 0$.

For $\mathcal{N} > x$ such that $x > \mathcal{N}_0(1/2)$, we then have

$$\mathsf{n}_I > \nu(x) > \frac{1}{2} \tag{3.5.9}$$

and

$$\rho \in [\rho_L(x), \rho_H(x)], \tag{3.5.10}$$

where

$$\rho_L(x) := 1 - \frac{1}{2\nu(x)+1} \quad \text{and} \quad \rho_H(x) := 1 + \frac{1}{2\nu(x)-1}. \tag{3.5.11}$$

We can make $\rho_L(x)$ and $\rho_H(x)$ arbitrarily close to 1 by making x large enough. More precisely,

$$\rho_L(x) \nearrow 1, \text{ and } \rho_H(x) \searrow 1 \text{ as } x \to \infty. \tag{3.5.12}$$

3.5.2 Rounding the number of regions in a collar

To model the second rounding step of the partition algorithm, we take note of the following two lemmas.

Lemma 3.5.3. *For $d > 1$ and $\mathcal{N} > 1$, with n, ϑ_i, $\vartheta_{F,i}$, y_i, m_i, a_i and $\mathcal{V}_R$ as per (3.2.5), (3.2.11), (3.2.8), (3.2.9), (3.2.10) and (3.2.1) respectively, we have*

$$\sum_{i=1}^{\mathsf{n}} \mathsf{y}_i = \mathcal{N} - 2 \tag{3.5.13}$$

and

$$\mathsf{m}_i = \mathsf{y}_i + \mathsf{a}_{i-1} - \mathsf{a}_i, \tag{3.5.14}$$

for $i \in \{1, \ldots, \mathsf{n}\}$;

$$\mathcal{V}(\vartheta_i) = \mathcal{V}(\vartheta_{F,i}) + \mathsf{a}_{i-1}\mathcal{V}_R, \tag{3.5.15}$$

for $i \in \{1, \ldots, \mathsf{n}+1\}$*; and*

$$\mathsf{m}_i = \frac{\mathcal{V}(\vartheta_{i+1}) - \mathcal{V}(\vartheta_i)}{\mathcal{V}_R}, \tag{3.5.16}$$

for $i \in \{1, \ldots, \mathsf{n}\}$.

Lemma 3.5.4. *Given a finite sequence* y *of length* n*, where*

$$\mathsf{y}_i \in \mathbb{R}, \quad \mathsf{y}_i \geqslant \frac{1}{2} \quad \textit{for} \quad i \in \{1, \ldots, \mathsf{n}\}, \quad \textit{and} \quad \sum_{i=1}^{\mathsf{n}} \mathsf{y}_i = \mathcal{N} - 2 \in \mathbb{N},$$

if we use (3.2.10) to define the sequences m *and* a*, then* m *and* a *have the following properties:*

$$\mathsf{m}_1 \in \mathbb{N}, \quad \mathsf{m}_i \in \mathbb{N}_0 \quad \textit{for} \quad i \in \{2, \ldots, \mathsf{n}\}, \tag{3.5.17}$$

$$\mathsf{a}_i \in \left[-\frac{1}{2}, \frac{1}{2}\right) \quad \textit{for} \quad i \in \{1, \ldots, \mathsf{n}\}, \tag{3.5.18}$$

$$\mathsf{a}_n = 0, \quad \sum_{i=1}^{\mathsf{n}} \mathsf{m}_i = \sum_{i=1}^{\mathsf{n}} \mathsf{y}_i. \tag{3.5.19}$$

If the sequence y is symmetric, then m and a have additional symmetry properties, which we examine here. These symmetry properties are not needed for the proof of Theorem 3.1.3, but are used to optimize the Matlab implementation of the partition algorithm.

Lemma 3.5.5. *Assuming the definitions of Lemma 3.5.4, define* $L := \lfloor \frac{\mathsf{n}}{2} \rfloor$.

If in addition to conditions of Lemma 3.5.4, the sequence y *is symmetric, that is,* $\mathsf{y}_i = \mathsf{y}_{\mathsf{n}-i+1}$ *for* $i \in \{1, \ldots, L\}$*, then the sequences* m *and* a *then have the following properties.*

1. *If* n *is even and* $\mathcal{N}$ *is odd then the sequence* m *is not symmetric.*
2. *If the sequence* m *is symmetric then the sequence* a *has* $\mathsf{a}_j = -\mathsf{a}_{\mathsf{n}-j}$ *for all* $j \in \{0, \ldots, L\}$.
3. *The sequence* m *is symmetric if and only if, for all* $i \in \{1, \ldots, L\}$*,* $\mathsf{a}_i \neq -\frac{1}{2}$.

Remarks. From Step 5 of the partition algorithm, we see that the sequence y is symmetric. Thus the only circumstances where m is not symmetric are when n is even and $\mathcal{N}$ is odd, and otherwise when some $\mathsf{a}_i = -\frac{1}{2}$. Since the sequence a is the result of a rounding process, in practice we rarely have any $\mathsf{a}_i = -\frac{1}{2}$ unless n is even and $\mathcal{N}$ is odd. Therefore the sequence m is symmetric often enough for this property to be useful in the optimization of the Matlab implementation of the partition algorithm.

3.5.3 Functions which model the regions in a collar

To make it easier to find bounds for functions which vary from zone to zone, such as y, m we define and use continuous analogs of these functions. This way, instead of having to find a bound for a function value over $\mathsf{n}+2$ points, where n varies with $\mathcal{N}$, we need only find a bound for a function over a fixed number of points and continuous intervals.

Motivated by Lemmas 3.5.3 and 3.5.4, we define

$$\mathcal{Y}(\vartheta) := \frac{\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta)}{\mathcal{V}_R}, \tag{3.5.20}$$

$$\mathcal{T}(\tau, \vartheta) := \Theta\big(\mathcal{V}(\vartheta) - \tau \mathcal{V}_R\big), \tag{3.5.21}$$

$$\mathcal{B}(\beta, \vartheta) := \Theta\big(\mathcal{V}(\vartheta + \delta_F) + \beta \mathcal{V}_R\big), \tag{3.5.22}$$

$$\mathcal{M}(\tau, \beta, \vartheta) := \mathcal{Y}(\vartheta) + \tau + \beta, \tag{3.5.23}$$

$$\Delta(\tau, \beta, \vartheta) := \mathcal{B}(\beta, \vartheta) - \mathcal{T}(\tau, \vartheta), \tag{3.5.24}$$

$$\mathcal{W}(\tau, \beta, \vartheta) := \max_{\xi \in [\mathcal{T}(\tau,\vartheta), \mathcal{B}(\beta,\vartheta)]} \sin \xi, \tag{3.5.25}$$

$$\mathcal{P}(\tau, \beta, \vartheta) := \mathcal{W}(\tau, \beta, \vartheta)\, \mathcal{M}(\tau, \beta, \vartheta)^{\frac{1}{1-d}}. \tag{3.5.26}$$

These functions have the following desirable properties.

Lemma 3.5.6. *For $i \in \{1, \ldots, \mathsf{n}\}$ we have*

$$\mathcal{Y}(\vartheta_{F,i}) = \mathsf{y}_i, \tag{3.5.27}$$

$$\mathcal{T}(-\mathsf{a}_{i-1}, \vartheta_{F,i}) = \vartheta_i, \tag{3.5.28}$$

$$\mathcal{B}(\mathsf{a}_i, \vartheta_{F,i}) = \vartheta_{i+1}, \tag{3.5.29}$$

$$\mathcal{M}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathsf{m}_i, \tag{3.5.30}$$

$$\Delta(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \delta_i, \tag{3.5.31}$$

$$\mathcal{W}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathsf{w}_i, \tag{3.5.32}$$

$$\mathcal{P}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathsf{p}_i. \tag{3.5.33}$$

3.5.4 Symmetries of the continuous analogs

Lemma 3.5.7. *The function $\mathcal{Y}$ satisfies*

$$\mathcal{Y}(\pi - \vartheta) = \mathcal{Y}(\vartheta - \delta_F). \tag{3.5.34}$$

Lemma 3.5.8. *The functions $\mathcal{T}$ and $\mathcal{B}$ satisfy the identities*

$$\mathcal{T}(\tau, \pi - \vartheta) = \pi - \mathcal{B}(\tau, \vartheta - \delta_F), \quad \textit{and} \tag{3.5.35}$$

$$\mathcal{B}(\beta, \pi - \vartheta) = \pi - \mathcal{T}(\beta, \vartheta - \delta_F). \tag{3.5.36}$$

Lemma 3.5.9. *For each $f \in \{\mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}\}$, the function f satisfies*

$$f(\tau, \beta, \pi - \vartheta) = f(\beta, \tau, \vartheta - \delta_F). \tag{3.5.37}$$

3.5.5 Feasible domains

For our feasible domain we therefore use the set $\mathbb{D}$, defined as follows.

Definition 3.5.10. *The* feasible domain $\mathbb{D}$ *is defined as*

$$\mathbb{D} := \mathbb{D}_t \cup \mathbb{D}_m \cup \mathbb{D}_b,$$

where

$$\mathbb{D}_t := \{(\tau, \beta, \vartheta) \mid \tau = 0, \beta \in \left[-\frac{1}{2}, \frac{1}{2}\right], \vartheta = \vartheta_c\}, \tag{3.5.38}$$

$$\mathbb{D}_m := \{(\tau, \beta, \vartheta) \mid \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right], \beta \in \left[-\frac{1}{2}, \frac{1}{2}\right], \vartheta \in [\vartheta_{F,2}, \pi - \vartheta_c - 2\delta_F]\}, \tag{3.5.39}$$

$$\mathbb{D}_b := \{(\tau, \beta, \vartheta) \mid \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right], \beta = 0, \vartheta = \pi - \vartheta_c - \delta_F\}. \tag{3.5.40}$$

We use closed intervals for τ and β because we need to support a number of symmetry properties which we will examine in Section 3.5.4 below.

Lemma 3.5.11. *Assume that $d > 1$ and that $\mathrm{EQ}(d-1)$ has diameter bound κ. Then for $\mathcal{N} > 2$, if we define*

$$\operatorname{maxdiam}(d, \mathcal{N}) := \max_{\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})} \operatorname{diam} \mathsf{R}, \tag{3.5.41}$$

then

$$\operatorname{maxdiam}(d, \mathcal{N}) \leqslant \sqrt{(\max_{\mathbb{D}} \Delta)^2 + \kappa^2 (\max_{\mathbb{D}} \mathcal{P})^2},$$

3.5.6 Symmetries of the feasible domain $\mathbb{D}$

We now show that we need only consider the northern hemisphere to obtain a valid bound for the diameter of a region of the recursive zonal equal area partition of $\mathbb{S}^d$. We first define the following subdomains of the feasible domain $\mathbb{D}$.

$$\mathbb{D}_+ := \left\{(\tau, \beta, \vartheta) \in \mathbb{D} \;\middle|\; \vartheta \leqslant \frac{\pi}{2} - \frac{\delta_F}{2}\right\}, \tag{3.5.42}$$

$$\mathbb{D}_- := \left\{(\tau, \beta, \vartheta) \in \mathbb{D} \;\middle|\; \vartheta > \frac{\pi}{2} - \frac{\delta_F}{2}\right\}, \tag{3.5.43}$$

$$\mathbb{D}_{m+} := \mathbb{D}_m \cap \mathbb{D}_+, \tag{3.5.44}$$

The following result then holds.

Lemma 3.5.12. *For $f \in \{\mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}\}$ and $(\tau, \beta, \vartheta) \in \mathbb{D}_-$, we can find $(\tau', \beta', \vartheta') \in \mathbb{D}_+$ such that $f(\tau', \beta', \vartheta') = f(\tau, \beta, \vartheta)$. In particular, if $(\tau, \beta, \vartheta) \in \mathbb{D}_b$, then $(\tau', \beta', \vartheta') \in \mathbb{D}_t$, and if $(\tau, \beta, \vartheta) \in \mathbb{D}_{m-}$, then $(\tau', \beta', \vartheta') \in \mathbb{D}_{m+}$.*

Corollary 3.5.13. *For $f \in \{\mathcal{M}, \Delta, \mathcal{W}, \mathcal{P}\}$,*

$$\max_{\mathbb{D}} f = \max_{\mathbb{D}_+} f.$$

3.5.7 Estimates

Recall from (3.2.2) and (3.2.1) that $\vartheta_c = \Theta\left(\frac{\omega_d}{\mathcal{N}}\right)$ and define

$$J_c(x) := \operatorname{sinc} \Theta\left(\frac{\omega_d}{x}\right). \tag{3.5.45}$$

As a result of (2.3.48), for $\mathcal{N} \geqslant x \geqslant 2$ we have

$$\vartheta_c \in [1, J_c(x)^{\frac{1-d}{d}}] \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \delta_I. \tag{3.5.46}$$

Using Lemma 2.3.18, we obtain the following upper bound for $\sin \vartheta_c$.

Lemma 3.5.14. *For $x \geqslant 2$,*

$$x^{\frac{1}{d}} \sin \Theta\left(\frac{\omega_d}{x}\right) \leqslant \left(\frac{\omega_d\ d}{\omega_{d-1}}\right)^{\frac{1}{d}}. \tag{3.5.47}$$

Therefore, for $\mathcal{N} \geqslant 2$,

$$\sin \vartheta_c \leqslant \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \delta_I. \tag{3.5.48}$$

Combining (3.5.45), (3.5.46) and (3.5.48) we have the estimate

$$\sin \vartheta_c \in [J_c(x), 1] \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \delta_I \tag{3.5.49}$$

for $\mathcal{N} \geqslant x \geqslant 2$.

Recalling (3.2.3), we also have

$$\delta_I = \omega_d^{\frac{1}{d}} \mathcal{N}^{-\frac{1}{d}}. \tag{3.5.50}$$

As another consequence of (3.5.46) we have the following estimate for the number of collars of $\mathrm{EQ}(d, \mathcal{N})$.

Lemma 3.5.15. *For $\mathcal{N} > \mathcal{N}_0(1/2)$, the number of collars n satisfies the estimate*

$$\mathsf{n} \leqslant \frac{\pi}{\delta_I}. \tag{3.5.51}$$

The definitions of the functions Δ and $\mathcal{P}$ and the definition of the feasible domain $\mathbb{D}$ depend on the fitting collar angle δ_F. Thus the proofs of Lemmas 3.5.25 and 3.5.26 need an estimate for δ_F.

Recall from (3.5.2) that $\delta_F = \rho \delta_I$. Therefore, from (3.5.10), for $\mathcal{N} > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.5.8) we have

$$\delta_F \in [\rho_L(x), \rho_H(x)] \delta_I. \tag{3.5.52}$$

We also need estimates for $\vartheta_{F,i}$, as defined by (3.2.8), and for $\sin \vartheta_{F,i}$ and $\mathcal{V}(\vartheta_{F,i})$.

Here and below, we generalize the definition of $\vartheta_{F,i}$, by defining

$$\vartheta_{F,\iota} := \vartheta_c + (\iota - 1)\delta_F, \tag{3.5.53}$$

for $\iota \in [1, \mathsf{n}+1]$.

For $\mathcal{N} > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.5.8), the estimates (3.5.46) and (3.5.52) now yield

$$\vartheta_{F,\iota} \in \left[\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + (\iota - 1)\rho_L(x), \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (\iota - 1)\rho_H(x) \right] \delta_I. \tag{3.5.54}$$

The estimates for $\sin\vartheta_{F,\iota}$ and $\mathcal{V}(\vartheta_{F,\iota})$ below assume that $\mathcal{N} > x > \mathcal{N}_0(1/2)$, where $\mathcal{N}_0$ is defined by (3.5.8), and the lower bounds for these estimates also assume that

$$\Theta\left(\frac{\omega_d}{x}\right) + (\iota-1)\rho_H(x)\left(\frac{\omega_d}{x}\right)^{\frac{1}{d}} \leqslant \frac{\pi}{2}. \tag{3.5.55}$$

If we define

$$J_{F,\iota}(x) := \operatorname{sinc}\left(\Theta\left(\frac{\omega_d}{x}\right) + (\iota-1)\rho_H(x)\left(\frac{\omega_d}{x}\right)^{\frac{1}{d}}\right), \tag{3.5.56}$$

then from (2.2.6) and (3.5.54) we have the estimate

$$\sin\vartheta_{F,\iota} \in \left[J_{F,\iota}(x)\left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + (\iota-1)\rho_L(x)\right), \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (\iota-1)\rho_H(x)\right]\delta_I \tag{3.5.57}$$

and from (2.3.47) we have the estimate

$$\mathcal{V}(\vartheta_{F,\iota}) \in [s_{L,\iota}(x), s_{H,\iota}(x)]\mathcal{V}_R, \tag{3.5.58}$$

where

$$s_{L,\iota}(x) := J_{F,\iota}(x)^{d-1}\left(1 + (\iota-1)\rho_L(x)\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d,$$
$$s_{H,\iota}(x) := \left(J_c(x)^{\frac{1-d}{d}} + (\iota-1)\rho_H(x)\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d.$$

If we define

$$s_\iota := \left(1 + (\iota-1)\left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d, \tag{3.5.59}$$

then, since $J_{F,\iota}(x) \nearrow 1$, $J_c(x) \nearrow 1$, $\rho_L(x) \nearrow 1$ and $\rho_H(x) \searrow 1$, as $x \to \infty$ we see that

$$s_{L,\iota}(x) \nearrow s_\iota \text{ and } s_{H,\iota}(x) \searrow s_\iota \tag{3.5.60}$$

as $x \to \infty$.

By making x large enough and ι small enough, we can ensure that (3.5.55) holds.

Lemma 3.5.16. *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), then (3.5.55) holds for*

$$\iota \in \left[1, \frac{13}{4}\right].$$

For the remainder of this chapter we use the abbreviation

$$\eta := \frac{1}{\sqrt{8\pi d}}. \tag{3.5.61}$$

The proofs of Lemmas 3.5.25 and 3.5.26 require the following results, which follow from Lemma 2.3.23.

Lemma 3.5.17. *There is an $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), such that*

$$J_{F,(1+\eta)}(x)^{d-1} \left(1 + \eta \ \rho_L(x) \ \left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d > \frac{3}{2}. \tag{3.5.62}$$

Lemma 3.5.18. *There is an $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), such that x satisfies (3.5.62) and also satisfies*

$$\rho_L(x) \ \omega_{d-1} \ J_{F,2}(x)^{d-1} \left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + \rho_L(x)\right)^{d-1} > 1. \tag{3.5.63}$$

The following result and its corollaries are also used in the proofs of Lemmas 3.5.25 and 3.5.26.

Lemma 3.5.19. *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x satisfies (3.5.62) then for $\mathcal{N} > x$ we have*

$$\mathcal{V}(\vartheta_c + \eta\delta_F) > \frac{3}{2}\mathcal{V}_R. \tag{3.5.64}$$

As a result of (3.5.64) we have

$$\mathcal{V}(\vartheta_c + \eta\delta_F) - \mathcal{V}(\vartheta_c) > \frac{\mathcal{V}_R}{2}. \tag{3.5.65}$$

Using Lemma 2.3.16 and the symmetries of the sine function, we have

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\left(\mathcal{V}(\vartheta + \eta\delta_F) - \mathcal{V}(\vartheta)\right) &= D\mathcal{V}(\vartheta + \eta\delta_F) - D\mathcal{V}(\vartheta)\\ &= \omega_{d-1}\left(\sin^{d-1}(\vartheta + \eta\delta_F) - \sin^{d-1}\vartheta\right)\\ &\geqslant 0 \quad \text{for } \vartheta \in \left(0, \frac{\pi}{2} - \eta\frac{\delta_F}{2}\right], \end{aligned} \tag{3.5.66}$$

with equality only when $\vartheta = \frac{\pi}{2} - \eta\frac{\delta_F}{2}$.

This results in the following corollary.

Corollary 3.5.20. *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x satisfies (3.5.62) then for $\mathcal{N} > x$ and $\vartheta \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$ we have*

$$\mathcal{V}(\vartheta + \eta\delta_F) - \mathcal{V}(\vartheta) > \frac{\mathcal{V}_R}{2}. \tag{3.5.67}$$

If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and $\mathcal{N} > x$ then $\mathsf{n} \geqslant 5$, so $\vartheta_{F,2} < \frac{\pi}{2}$. Since $8\pi d \geqslant 16\pi > 49$, we therefore have

$$\eta\delta_F < \frac{\delta_F}{7}. \tag{3.5.68}$$

The convexity of $\mathcal{V}$ as per Lemma 2.3.16 together with (3.5.69) and (3.5.68) then yield the following results.

Corollary 3.5.21. *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x satisfies (3.5.62) then for $\mathcal{N} > x$ and $\vartheta \in [\vartheta_c, \pi - \vartheta_c - \delta_F]$ we have*

$$\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta) > \frac{7}{2}\mathcal{V}_R. \tag{3.5.69}$$

In particular,

$$\mathcal{V}(\vartheta_{F,2}) > \frac{9}{2}\mathcal{V}_R. \tag{3.5.70}$$

By (2.3.41) and (2.3.42) we also have

$$\mathcal{V}(\vartheta_{F,2}) - \mathcal{V}(\vartheta_c) \leqslant \delta_F \ \omega_{d-1} \sin^{d-1} \vartheta_{F,2},$$

and we have the following result.

Corollary 3.5.22. *If $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x satisfies (3.5.62) then for $\mathcal{N} > x$ we have*

$$\delta_F \ \omega_{d-1} \sin^{d-1} \vartheta_{F,2} > \frac{7}{2}\mathcal{V}_R. \tag{3.5.71}$$

The upper bound for $\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta)$ is a little easier to analyze than the lower bound. From (2.3.36), for $\vartheta \in [0, \pi - \delta_F]$ we have the crude upper bound

$$\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta) = \omega_{d-1} \int_{\vartheta}^{\vartheta+\delta_F} \sin^{d-1} \xi \ d\xi \leqslant \omega_{d-1} \ \delta_F. \tag{3.5.72}$$

While crude, this bound is sufficient to prove the following estimates. Firstly, from (3.5.20) and (3.5.72) we immediately have for $\mathcal{Y}(\vartheta)$ for $\vartheta \in [0, \pi - \delta_F]$ the estimate

$$\mathcal{Y}(\vartheta) \leqslant \frac{\omega_{d-1}}{\mathcal{V}_R} \ \delta_F.$$

Together with (3.2.1), (3.2.3) and (3.5.2), this gives us

$$\mathcal{Y}(\vartheta) \leqslant \rho \ \omega_{d-1} \ \delta_I^{1-d} = \rho \ \omega_{d-1} \ \omega_d^{\frac{1-d}{d}} \ \mathcal{N}^{\frac{d-1}{d}}. \tag{3.5.73}$$

Secondly, we have a crude estimate for the maximum number of regions in zone i.

Lemma 3.5.23. *For $i \in \{0, \ldots, \mathcal{N}+1\}$, the number of regions m_i in zone i of* $\mathrm{EQ}(d,\mathcal{N})$ *satisfies the upper bound*

$$\mathsf{m}_i \leqslant \begin{cases} 1, & \mathcal{N} \leqslant 2, \\ \mathcal{N} - 2, & \mathcal{N} \in [3, \mathcal{N}_0(1/2))\,, \\ \rho_H(x)\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}}\ \mathcal{N}^{\frac{d-1}{d}} + 1, & \mathcal{N} \geqslant x \geqslant \mathcal{N}_0(1/2). \end{cases}$$

As a result, we have the estimate

$$\mathsf{m}_i \leqslant \mathsf{m}_{\ominus} := \mathsf{C}_{\ominus,d}\ \mathcal{N}^{\frac{d-1}{d}}, \tag{3.5.74}$$

where

$$\mathsf{C}_{\ominus,d} := \max\left(\mathcal{N}_0(1/2)^{\frac{1}{d}} - 2\ \mathcal{N}_0(1/2)^{\frac{1-d}{d}}, \rho_H\big(\mathcal{N}_0(1/2)\big)\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}} + 1\right). \tag{3.5.75}$$

3.5.8 Bounds

As a consequence of Theorem 3.3.2 (the isodiametric inequality), the following result on diameter bounds for the polar cap is not needed for the proof of Theorem 3.1.3. It is included for completeness, and for comparison to the Feige–Schechtman bound to be examined below.

As an immediate consequence of Lemma 3.5.14 we have the following upper bound for the diameter of a polar cap of $\mathrm{EQ}(d,\mathcal{N})$.

Lemma 3.5.24. *For $d > 1$ and $\mathcal{N} \geqslant 2$, the diameter of each polar cap of* $\mathrm{EQ}(d,\mathcal{N})$ *is bounded above by $K_c \mathcal{N}^{-\frac{1}{d}}$, where*

$$K_c := 2\left(\frac{\omega_d\ d}{\omega_{d-1}}\right)^{\frac{1}{d}}. \tag{3.5.76}$$

The following two bounds are used in the proof of Theorem 3.1.3.

Lemma 3.5.25. *For $d > 1$, there is a positive constant $N_\Delta \in \mathbb{N}$ and a monotonic decreasing positive real function K_Δ such that for each partition* $\mathrm{EQ}(d,\mathcal{N})$ *with*

$\mathcal{N} > x \geqslant N_\Delta$,

$$\max_{\mathbb{D}} \Delta \leqslant K_\Delta(x)\mathcal{N}^{-\frac{1}{d}}.$$

Lemma 3.5.26. *For $d > 1$, there is a positive constant $N_P \in \mathbb{N}$ and a monotonic decreasing positive real function C_P such that for each partition $\mathrm{EQ}(d, \mathcal{N})$ with $\mathcal{N} > x \geqslant N_P$,*

$$\max_{\mathbb{D}} \mathcal{P} \leqslant \mathsf{C}_P(x)\mathcal{N}^{-\frac{1}{d}}.$$

3.6 Proofs of main theorems

We first examine the equal area property.

Proof of Theorem 3.1.2.

The theorem is true for $d = 1$, since the recursive zonal equal area partition algorithm partitions the circle $\mathbb{S}^1$ into $\mathcal{N}$ equal segments.

We now assume that $d > 1$. At Step 7 of the partition algorithm, we define zone i to be $Z(\vartheta_i, \vartheta_{i+1})$. Using (3.2.11), the area of each polar cap is $\mathcal{V}_R$ and, for $i \in \{1, \dots, \mathsf{n}\}$, the area of collar i is

$$\sigma\big(Z(\vartheta_i, \vartheta_{i+1})\big) = \mathsf{m}_i \mathcal{V}_R.$$

From Lemmas 3.5.3 and 3.5.4, we know that m_i is a positive integer. At Step 3.2.3 of the recursive zonal equal area partition algorithm, we recursively use $\mathrm{EQ}(d-1, \mathsf{m}_i)$ to partition each collar into m_i regions. Therefore, by induction, each region of the partition $\mathrm{EQ}(d, \mathcal{N})$ has area $\mathcal{V}_R$. □

We now examine the diameter property.

Proof of Theorem 3.1.3.

The theorem is true for $d = 1$, with $\mathrm{EQ}(1)$ having diameter bound $K_1 = 2\pi$, since the recursive zonal equal area partition algorithm partitions the circle $\mathbb{S}^1$ into $\mathcal{N}$

equal segments, each of arc length $2\pi/\mathcal{N}$, and therefore each segment has diameter less than $2\pi/\mathcal{N}$.

Now assume that $d > 1$ and $\mathcal{N} > 2$. We know from Lemma 3.5.11 that

$$\operatorname{maxdiam}(d, \mathcal{N}) \leqslant \sqrt{\left(\max_{\mathbb{D}} \Delta\right)^2 + \kappa^2 \left(\max_{\mathbb{D}} \mathcal{P}\right)^2}.$$

From Lemma 3.5.25, we know that there is a positive constant $N_\Delta \in \mathbb{N}$ and a monotonic decreasing positive real function K_Δ such that for each partition $\mathrm{EQ}(d, \mathcal{N})$ with $\mathcal{N} > x \geqslant N_\Delta$,

$$\max_{\mathbb{D}} \Delta \leqslant K_\Delta(x) \mathcal{N}^{-\frac{1}{d}}.$$

From Lemma 3.5.26, we know that there is a positive constant $N_P \in \mathbb{N}$ and a monotonic decreasing positive real function C_P such that for each partition $\mathrm{EQ}(d, \mathcal{N})$ with $\mathcal{N} > x \geqslant N_P$,

$$\max_{\mathbb{D}} \mathcal{P} \leqslant \mathsf{C}_P(x) \mathcal{N}^{-\frac{1}{d}}.$$

Define

$$N_H := \max(N_\Delta, N_P). \tag{3.6.1}$$

Assuming that $\mathrm{EQ}(d-1)$ is diameter bounded, with diameter bound κ, then for $\mathcal{N} > N_H$, we have $\operatorname{maxdiam}(d, \mathcal{N}) \leqslant K_H \mathcal{N}^{-\frac{1}{d}}$, where

$$K_H := \sqrt{K_\Delta(N_H)^2 + \kappa^2 \mathsf{C}_P(N_H)^2}. \tag{3.6.2}$$

For $d > 1$ and $\mathcal{N} \leqslant N_H$, we note that the diameter of $\mathbb{S}^d$ is 2, and so the diameter of any region is bounded by 2. Therefore for $\mathcal{N} \leqslant N_H$, $\operatorname{maxdiam}(d, \mathcal{N}) \leqslant K_L \mathcal{N}^{-\frac{1}{d}}$,

where

$$K_L := 2 N_H^{\frac{1}{d}}. \tag{3.6.3}$$

Finally, we see by induction that for $d > 1$, $\operatorname{maxdiam}(d, \mathcal{N}) \leqslant K_d \mathcal{N}^{-\frac{1}{d}}$, where

$$K_d := \max(K_L, K_H). \tag{3.6.4}$$

□

3.7 A per-region bound on diameter

The following bound is not needed for the proof of Theorem 3.1.3, but is useful in checking the calculation of the diameters of individual regions.

Definition 3.7.1. *The region diameter bound function* db *is defined on the regions of a partition* $\operatorname{EQ}(d, \mathcal{N})$ *as follows.*

For the whole sphere $\mathbb{S}^d$*,*

$$\operatorname{db} \mathbb{S}^d := 2.$$

For a region R *contained in* $\operatorname{EQ}(1, \mathcal{N})$*,*

$$\operatorname{db} \mathsf{R} := \Upsilon\left(\frac{2\,\pi}{\mathcal{N}}\right),$$

where Υ *is defined by (2.3.2).*

For $d > 1$*, for a spherical cap* R *with spherical radius* ϑ_c*,*

$$\operatorname{db} \mathsf{R} := 2\, \sin \vartheta_c.$$

For $d > 1$, for a region R contained in collar $i \in \{1, \ldots, \mathsf{n}\}$ of a partition $\mathrm{EQ}(d,\mathcal{N})$ with n collars,

$$\mathrm{db}\,\mathsf{R} := \sqrt{\Upsilon(\delta_i)^2 + \mathsf{w}_i^2(\mathrm{db}\,\Pi\,\mathsf{R})^2},$$

where $\Pi\,\mathsf{R}$ is defined by (3.3.1).

Theorem 3.7.2. *For any region $\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})$,*

$$\mathrm{diam}\,\mathsf{R} \leqslant \mathrm{db}\,\mathsf{R}.$$

3.8 Proof of Stolarsky's assertion

Feige and Schechtman's construction yields the following upper bound on the smallest maximum diameter of an equal area partition of $\mathbb{S}^d$.

Lemma 3.8.1. *[54, Lemma 21, p. 430-431]*

For $d > 1$, $\mathcal{N} > 2$, there is a partition $FS(d,\mathcal{N})$ of the unit sphere $\mathbb{S}^d$ into $\mathcal{N}$ regions, with each region $\mathsf{R} \in FS(d,\mathcal{N})$ having area $\omega_d/\mathcal{N}$ and Euclidean diameter bounded above by

$$\mathrm{diam}\,\mathsf{R} \leqslant \Upsilon\left(\min(\pi, 8\vartheta_c)\right), \tag{3.8.1}$$

with Υ defined by (2.3.2) and ϑ_c defined by (3.2.2).

We now use the Feige–Schechtman construction to prove Stolarsky's assertion.

Theorem 3.8.2. *[147, p. 581] For each $d > 0$, there is a constant c_d such that for all $\mathcal{N} > 0$, there is a partition of the unit sphere $\mathbb{S}^d$ into $\mathcal{N}$ regions, with each region having area $\omega_d/\mathcal{N}$ and diameter at most $c_d\mathcal{N}^{-\frac{1}{d}}$.*

Proof. For $d = 1$, we partition the circle into equal segments and the proof is as per the proof of Theorem 3.1.3.

For $d > 1$ and $\mathcal{N} = 1$, there is one region of diameter $2 = 2\mathcal{N}^{-\frac{1}{d}}$. For $d > 1$ and $\mathcal{N} = 2$, there are two regions, each of diameter $2 = 2^{\frac{d+1}{d}}\mathcal{N}^{-\frac{1}{d}}$.

Otherwise we use Lemma 3.8.1 and the estimates (3.5.46) and (3.5.48). Define

$$N_{FS} = \frac{\omega_d}{\mathcal{V}\left(\frac{\pi}{8}\right)}. \tag{3.8.2}$$

Then for $\mathcal{N} \geqslant N_{FS}$,

$$\vartheta_c = \Theta\left(\frac{\omega_d}{\mathcal{N}}\right) \leqslant \frac{\pi}{8}, \tag{3.8.3}$$

with equality only when $\mathcal{N} = N_{FS}$. Therefore, for $\mathcal{N} \geqslant N_{FS}$, Lemmas 3.5.14 and 3.8.1 give us

$$\max_{\mathsf{R} \in FS(d,\mathcal{N})} \operatorname{diam} \mathsf{R} \leqslant 2 \ \sin 4\vartheta_c < 8 \ \sin \vartheta_c < K_{FS} \mathcal{N}^{-\frac{1}{d}},$$

where

$$K_{FS} := 8 \left(\frac{\omega_d \ d}{\omega_{d-1}}\right)^{\frac{1}{d}}. \tag{3.8.4}$$

For $2 < \mathcal{N} < N_{FS}$, we have

$$\operatorname{maxdiam} FS(d,\mathcal{N}) \leqslant 2 = 2 \ \mathcal{N}^{\frac{1}{d}} \mathcal{N}^{-\frac{1}{d}} < 2 \ N_{FS}^{\frac{1}{d}} \mathcal{N}^{-\frac{1}{d}}.$$

Let $K_{FSL} := 2 \ N_{FS}^{\frac{1}{d}}$. Using (2.3.51) we have

$$\mathcal{V}\left(\frac{\pi}{8}\right) \geqslant \frac{1}{\operatorname{sinc} \frac{\pi}{8}} \frac{\omega_{d-1}}{d} \ \sin^d \frac{\pi}{8} > \frac{\omega_{d-1}}{d} \ \sin^d \frac{\pi}{8}.$$

We also have $\sin \frac{\pi}{8} > \frac{1}{4}$ so that

$$\mathcal{V}\left(\frac{\pi}{8}\right) > \frac{\omega_{d-1}}{4^d \ d}.$$

Therefore

$$N_{FS} = \frac{\omega_d}{\mathcal{V}\left(\frac{\pi}{8}\right)} < 4^d \ \frac{\omega_d \ d}{\omega_{d-1}},$$

in other words,

$$K_{FSL}^d = 2^d \ N_{FS} < 8^d \ \frac{\omega_d \ d}{\omega_{d-1}} = K_{FS}^d.$$

We therefore have $K_{FSL} < K_{FS}$.

For the case $\mathcal{N} = 2$, from (2.3.52) we obtain

$$2^{d+1} < 8^d \ \sqrt{2\pi d} < 8^d \ \frac{\omega_d \ d}{\omega_{d-1}} = K_{FS}^d.$$

Therefore Theorem 3.8.2 is satisfied by $c_d = K_{FS}$. □

Remarks. The Feige–Schechtman constant K_{FS} thus provides an upper bound for the minimum constant for the diameter bound of an equal area partition of $\mathbb{S}^d$.

Theorems 3.1.2 and 3.1.3 yield an alternate proof of Theorem 3.8.2, with $c_d = K_d$.

3.9 Implementation

The Recursive Zonal Equal Area (EQ) Sphere Partitioning Toolbox is a suite of Matlab [154] functions. These functions are intended for use in exploring different aspects of EQ sphere partitioning.

The functions are grouped into the following groups of tasks:

1. Create EQ partitions
2. Find properties of EQ partitions
3. Find properties of EQ point sets
4. Produce illustrations
5. Test the toolbox

6. Perform some utility function

An EQ point set is the set of centre points of the regions of an EQ partition. Each region is defined as a product of intervals in spherical polar coordinates. The centre point of a region is defined via the centre points of each interval, with the exception of spherical caps and their descendants, where the centre point is defined using the centre of the spherical cap.

The toolbox has been tested with Matlab versions 6.5 and 7.0.1 on Linux, and 6.5.1 on Windows.

3.9.1 Implementation of the functions $\mathcal{V}$ and Θ

For $d \leqslant 2$, the area function $\mathcal{V}_d(\theta)$ uses the closed solution to the integral (2.3.36), and for $d > 3$ the area function uses the Matlab [154] function `BETAINC` to evaluate the regularized incomplete Beta function I of Lemma 2.3.15. For $d = 3$ the area function uses the closed solution for $\theta \in [\pi/6, 5\pi/6]$ and otherwise uses `BETAINC.`

The inverse function $\Theta_d(v)$ uses the closed solution to the inverse for $d \leqslant 2$, and otherwise uses the Matlab [154] function `FZERO` to find the solution. This loses some accuracy for area arguments v near zero. In future, the inverse function may instead be based on an implementation of the inverse Beta distribution algorithm of Abernathy and Smith [1].

3.9.2 Limitations

Ultimately, the Matlab code is limited by the speed of the processor and the amount of memory available.

Any function which has `dim` as a parameter will work for any integer `dim` $\geqslant 1$.

Any function which takes `N` as an argument will work with any positive integer value of `N` , but for very large `N` , the function may be slow, or may consume large amounts of memory.

3.10 Numerical results

3.10.1 Maximum diameters of regions

Figures 3.4, 3.5 and 3.6 are log-log plots corresponding to the recursive zonal equal area partitions of $\mathbb{S}^d$ for $d = 2$, $d = 3$ and $d = 4$ respectively. For each partition $\mathrm{EQ}(d, \mathcal{N})$, for $\mathcal{N}$ from 1 to 100 000, each figure shows the coefficients corresponding to the maximum per-region upper bound on diameter, as per Definition 3.7.1, depicted as red dots, and the maximum vertex diameter, depicted as blue + signs. Each of these coefficients is obtained by multiplying the corresponding diameter bound by $\mathcal{N}^{\frac{1}{d}}$.

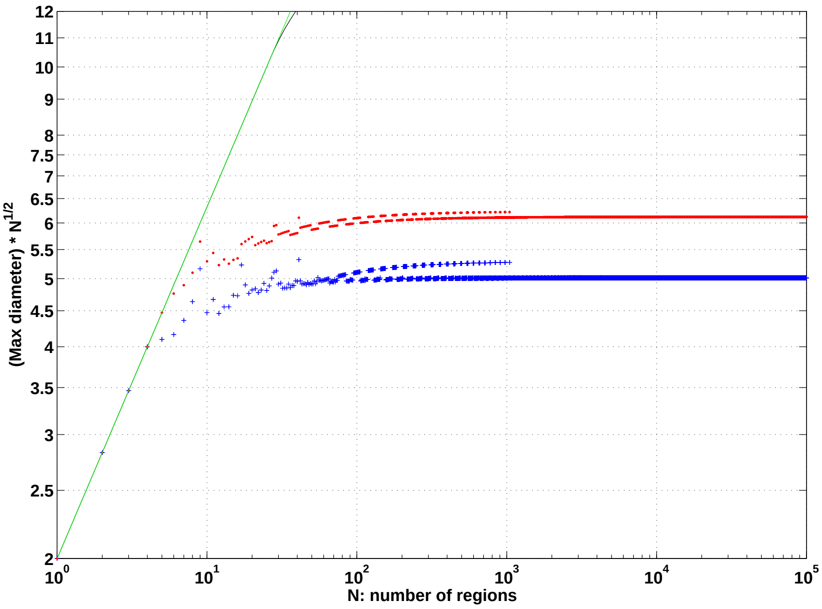

Figure 3.4: Maximum diameters of $\mathrm{EQ}(2, \mathcal{N})$ (log-log scale)

The vertex diameter of a region is the maximum distance between pseudo-vertices of a region, except where a region spans 2π in longitude, in which case one of each pair of coincident pseudo-vertices is replaced by a point with the same colatitudes and a longitude increased by π. For low dimensions and for regions which do not straddle the equator, the vertex diameter provides a good lower bound on the diameter.

Only the upper and lower bounds on the maximum diameter are plotted, rather than the maximum diameter itself. This is because, for each region of each partition,

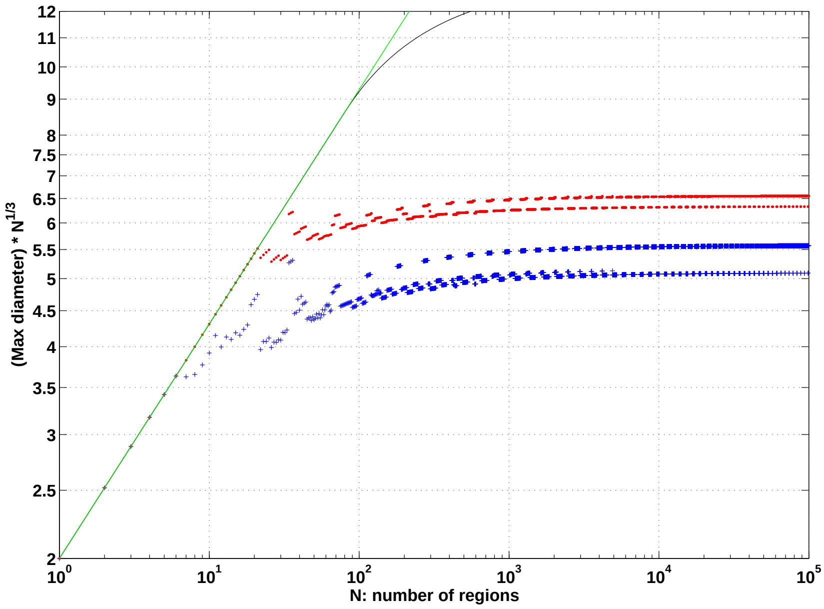

Figure 3.5: Maximum diameters of EQ$(3, \mathcal{N})$ (log-log scale)

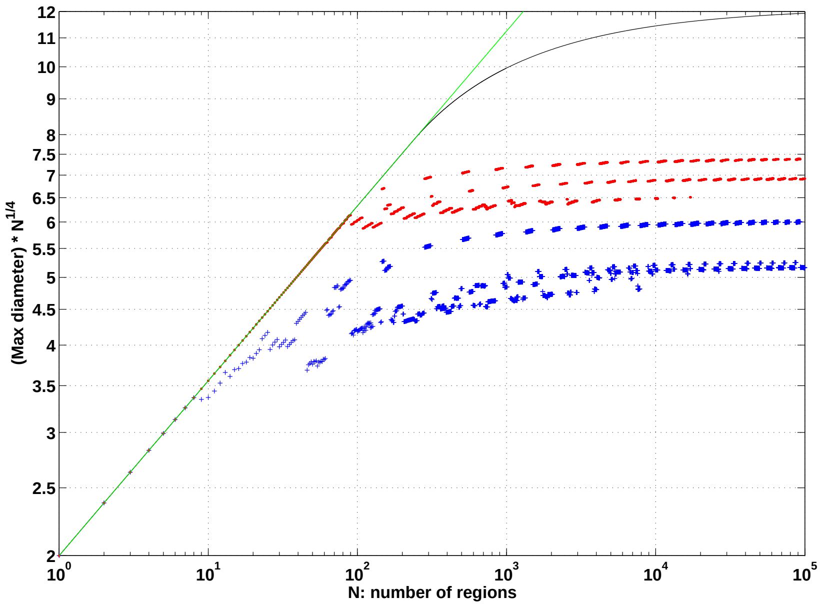

Figure 3.6: Maximum diameters of EQ$(4, \mathcal{N})$ (log-log scale)

the diameter is the solution of a constrained nonlinear optimization problem. It would therefore take quite a long time to calculate the maximum diameter of every partition for $\mathcal{N}$ from 1 to 100 000.

The black curve on each figure is the Feige-Schechtman bound (3.8.1). On each figure, this curve joins a straight line for which the maximum diameter of a region is 2.

Figures 3.4, 3.5 and 3.6 show that for $\mathcal{N} \leqslant 100\,000$, we have $\operatorname{maxdiam}(2,\mathcal{N})\mathcal{N}^{\frac{1}{2}} < 6.5$, $\operatorname{maxdiam}(3,\mathcal{N})\mathcal{N}^{1/3} < 7$ and $\operatorname{maxdiam}(4,\mathcal{N})\mathcal{N}^{1/4} < 7.5$ respectively. Figure 3.7 is a log-log plot corresponding to the partitions $\mathrm{EQ}(d, 2^k)$, for d from 2 to 8, for k from 1 to 20. The figure shows the coefficient obtained by multiplying $\mathcal{N}^{\frac{1}{d}}$ by the maximum per-region upper bound on diameter, as per Definition 3.7.1, depicted as red dots, and by the maximum vertex diameter, depicted as blue + signs. For the cases shown, we have $2^{\frac{k}{d}}\ \operatorname{maxdiam}(d, 2^k) < 8$.

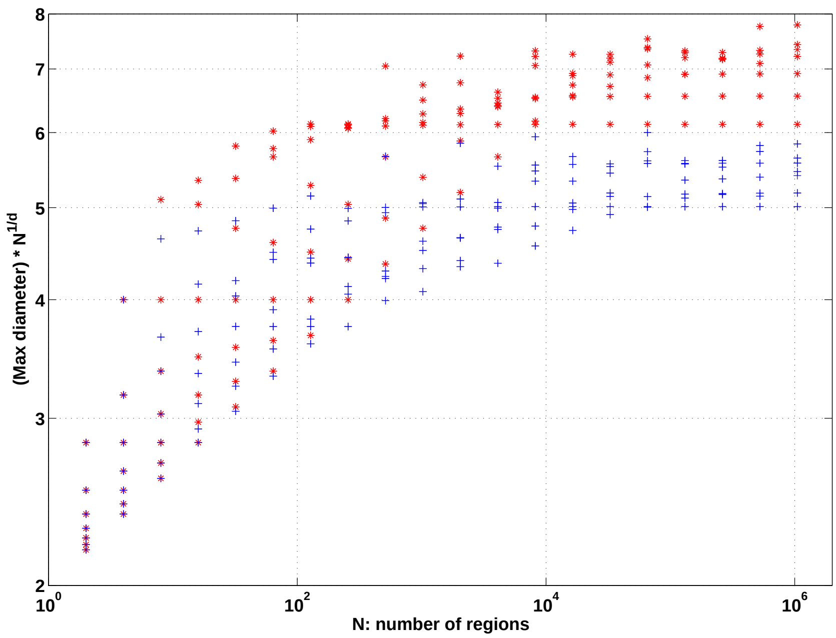

Figure 3.7: Maximum diameters of $\mathrm{EQ}(d,\mathcal{N})$, d from 2 to 8 (log-log scale)

3.10.2 Running time

To benchmark the speed of the partition algorithm, the function `eq_regions`$(d,\mathcal{N})$ was run for d from 1 to 11 and $\mathcal{N}$ from 2 to $2^{22} = 4194304$, in successive powers of 2, on a 2 GHz AMD Opteron processor, using Matlab 7.01 [154]. The benchmark was repeated a total of three times. For d from 2 to 11 and $\mathcal{N}$ from 8 to 2^{22}, the running time t was approximately

$$t(d,\mathcal{N}) = (0.24 \pm 0.04)\ d^{1.90\pm0.07}\ \mathcal{N}^{0.60\pm0.01}\ \text{ms},$$

with the error bounds having 95% confidence level. Thus for this range of d and $\mathcal{N}$, the running time of the partition algorithm is approximately $\mathrm{O}(\mathcal{N}^{0.6})$, which is faster than linear in $\mathcal{N}$.

3.11 Remaining proofs

3.11.1 The regions of a recursive zonal equal area partition

The following lemmas show that the diameter of a region of an EQ partition is attained by the maximum Euclidean distance between two points of the region. The proofs are entirely elementary – in fact almost obvious – and are included only for the sake of completeness.

Lemma 3.11.1. *Each* $\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})$ *is closed in the topology of* $\mathbb{S}^d$ *induced by the Euclidean metric in* $\mathbb{R}^{d+1}$*.*

Proof. Since, from (3.2.12) each $\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})$ is described as the product of closed intervals in spherical polar coordinates, R is closed in the topology of $\mathbb{S}^d$ induced by the Euclidean metric in $\mathbb{R}^{d+1}$. $\square$

Lemma 3.11.2. *For each* $\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})$*, there are two points* $\mathbf{a}, \mathbf{b} \in \mathsf{R}$ *such that the diameter of* R *is* $\|\mathbf{a} - \mathbf{b}\|$*.*

Proof. The diameter of R is the supremum of the Euclidean distance in $\mathbb{R}^{d+1}$ between pairs of points of $\mathsf{R} \in \mathbb{S}^d$.

Since, as a result of Lemma 3.11.1, R is closed, the diameter of R is the maximum Euclidean distance in $\mathbb{R}^{d+1}$ between pairs of points of R. That is, there are two points $\mathbf{a}, \mathbf{b} \in \mathsf{R}$ such that the diameter of R is $\|\mathbf{a} - \mathbf{b}\|$. $\square$

The proof of Lemma 3.3.1 depends on the following elementary results.

Lemma 3.11.3. *For any closed subset* R *of* $\mathbb{S}^d$*, either*

- *the diameter of* R *is 2 and* R *contains a pair of antipodal points, or*
- *there are two points,* $\mathbf{p}, \mathbf{q} \in \partial\mathsf{R}$*, the boundary of* R*, such that the diameter of* R *is the Euclidean distance* $\|\mathbf{p} - \mathbf{q}\|$*.*

Proof. The diameter of R is the maximum of the Euclidean distance in $\mathbb{R}^{d+1}$ between pairs of points of $\mathsf{R} \in \mathbb{S}^d$, since R is closed. That is, there are two points $\mathbf{p}, \mathbf{q} \in \mathsf{R}$ such that the diameter of R is $\|\mathbf{p} - \mathbf{q}\|$.

Now consider any two points, $\mathbf{a}, \mathbf{b} \in \mathsf{R}$. If $\mathbf{a}$ is an interior point of R, then either $\|\mathbf{a} - \mathbf{b}\| = 2$, which is the maximum possible distance, and $\mathbf{a}$ and $\mathbf{b}$ are antipodal, or there is a point $\mathbf{c}$ of R in a neighbourhood of $\mathbf{a}$ with $\|\mathbf{c} - \mathbf{b}\| > \|\mathbf{a} - \mathbf{b}\|$.

To see this, take a geodesic between $\mathbf{a}$ and $\mathbf{b}$ and extend it through any neighbourhood of $\mathbf{b}$ which is contained in R. Now recall from Lemma 2.3.2 that Euclidean distance is a monotonic increasing function of spherical distance, and that spherical distance is the same as geodesic arc length, up to π. □

Corollary 3.11.4. *The diameter of an arc C of $\mathbb{S}^1$, with end points $\mathbf{a}$ and $\mathbf{b}$, is the Euclidean distance between $\mathbf{a}$ and $\mathbf{b}$, if the arc length of C is less than π. Otherwise, the Euclidean diameter of C is 2.*

Proof of Lemma 3.3.1.

Since we have $d > 1$ and $\mathcal{N} > 1$, we know that the EQ algorithm yields two polar caps, both with spherical radius ϑ_c, and therefore we know that $\vartheta_c \leqslant \frac{\pi}{2}$.

Let R be the North polar cap with colatitude ϑ_c. From Lemma 3.11.3, we see that either R contains a pair of antipodal points or there are points $\mathbf{p}, \mathbf{q} \in \partial\mathsf{R}$ such that the diameter of R is $\|\mathbf{p} - \mathbf{q}\|$.

Since $\vartheta_c \leqslant \frac{\pi}{2}$, then any point $\mathbf{a} \in \mathsf{R}$ is of the form

$$\mathbf{a} = \odot(s_1, \ldots, \vartheta) = (x_1, \ldots, \cos\vartheta),$$

where $\vartheta < \vartheta_c \leqslant \frac{\pi}{2}$. The antipodal point, $-\mathbf{a}$ is therefore $-\mathbf{a} = (-x_1, \ldots, -\cos\vartheta)$. But we know that $-\cos\vartheta = \cos(\pi - \vartheta)$, but then $\pi - \vartheta > \pi - \vartheta_c \geqslant \frac{\pi}{2}$.

If $\vartheta_c < \frac{\pi}{2}$, then $-\mathbf{a}$ is not contained in R, and R does not contain pairs of antipodal points. If $\vartheta_c = \frac{\pi}{2}$, then R contains pairs of antipodal points, which are all boundary points. So there must be points $\mathbf{p}, \mathbf{q} \in \partial\mathsf{R}$ such that the diameter of R is $\|\mathbf{p} - \mathbf{q}\|$.

Now recall that $\partial \mathsf{R}$ is the parallel at colatitude ϑ_c, which is a small sphere of Euclidean radius $\sin \vartheta_c$ and Euclidean diameter $2 \sin \vartheta_c$.

The analysis of the South polar cap is almost identical. $\square$

Proof of Lemma 3.3.4.

For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{d+1}$ we have

$$\begin{aligned}\|\mathbf{a}-\mathbf{c}\|^2 + \|\mathbf{c}-\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{c}) \cdot (\mathbf{a}-\mathbf{c}) + (\mathbf{c}-\mathbf{b}) \cdot (\mathbf{c}-\mathbf{b}) - (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) \\ &= 2\,\mathbf{a} \cdot \mathbf{b} - 2\,\mathbf{a} \cdot \mathbf{c} - 2\,\mathbf{c} \cdot \mathbf{b} + 2\,\mathbf{c} \cdot \mathbf{c} = 2\,(\mathbf{a}-\mathbf{c}) \cdot (\mathbf{b}-\mathbf{c}).\end{aligned}$$

We therefore prove Lemma 3.3.4 by proving that $(\mathbf{a}-\mathbf{c}) \cdot (\mathbf{b}-\mathbf{c}) \geqslant 0$.

First, note that rotations of $\mathbb{S}^d$ are isometries and therefore without loss of generality we may rotate the triangle $\mathbf{acb}$ to make calculation more convenient. Now note that we can apply a single $\mathbb{S}^{d-1}$ rotation to $\mathbb{S}^d$ while keeping the $\mathbb{S}^d$ colatitude fixed. Therefore we can assume that

$$\mathbf{a} = \odot(0, \ldots, 0, 0, A), \quad \mathbf{b} = \odot(0, \ldots, 0, C, B), \quad \mathbf{c} = \odot(0, \ldots, 0, 0, B).$$

In Cartesian coordinates, for $d > 3$, we obtain

$$\begin{aligned}\mathbf{a} &= (0, \ldots, 0, 0, \sin A, \cos A), \\ \mathbf{b} &= (0, \ldots, 0, \sin B\, \sin C, \sin B\, \cos C, \cos B), \\ \mathbf{c} &= (0, \ldots, 0, 0, \sin B, \cos B).\end{aligned}$$

Due to an unfortunate feature of the conventional mapping from spherical to Cartesian coordinates, for $\mathbb{S}^2 \subset \mathbb{R}^3$, we obtain

$$\mathbf{a} = (\sin A, 0, \cos A), \quad \mathbf{b} = (\sin B\, \cos C, \sin B\, \sin C, \cos B), \quad \mathbf{c} = (\sin B, 0, \cos B),$$

and for $\mathbb{S}^3 \subset \mathbb{R}^4$, we obtain

$$\mathbf{a} = (0, 0, \sin A, \cos A), \quad \mathbf{b} = (\sin B\ \sin C, 0, \sin B\ \cos C, \cos B), \quad \mathbf{c} = (0, 0, \sin B, \cos B).$$

In all three cases, we obtain

$$\begin{aligned}(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) &= (\sin A - \sin B)(\sin B\ \cos C - \sin B) + (0)(\sin B\ \sin C) + (\cos A - \cos B)(0) \\ &= (\sin B - \sin A)\ (1 - \cos C)\ \sin B \geqslant 0.\end{aligned}$$

□

To prove Lemma 3.3.5 we use the following results.

Lemma 3.11.5. *Let* $\mathbf{a}$, $\mathbf{c}$ *be points of region* R *in collar* i *of* $\mathrm{EQ}(d, \mathcal{N})$, *which additionally satisfy (3.3.2) with* $\sin B \geqslant \sin A$, *that is,*

$$\begin{gathered}\mathsf{R} = \mathcal{R}\left((\tau_1, \ldots, \tau_{d-1}, \vartheta_i), (\upsilon_1, \ldots, \upsilon_{d-1}, \vartheta_{i+1})\right), \\ \mathbf{a} = \odot(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, A), \quad \mathbf{c} = \odot(\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, B), \\ \alpha_k \in [\tau_k, \upsilon_k], k \in \{1, \ldots, d-1\}, \quad A, B \in [\vartheta_i, \vartheta_{i+1}], \quad \sin B \geqslant \sin A.\end{gathered}$$

We then have $\|\mathbf{a} - \mathbf{c}\| \leqslant \Upsilon(\delta_i) < \delta_i$, *where* δ_i *is given by (3.3.3).*

Proof. Since $\mathbf{a}$ and $\mathbf{c}$ differ only in colatitude we have

$$\mathsf{s}(\mathbf{a}, \mathbf{c}) = |B - A| \leqslant \vartheta_{i+1} - \vartheta_i = \delta_i.$$

Using Lemma 2.3.2 we note that the function Υ increases monotonically with spherical distance, and for all $\theta \in (0, \pi]$ we have $\Upsilon(\theta) < \theta$. Therefore

$$\|\mathbf{a} - \mathbf{c}\| = \Upsilon\left(\mathsf{s}(\mathbf{a}, \mathbf{c})\right) < \Upsilon(\delta_i) < \delta_i.$$

□

Lemma 3.11.6. *Let* $\mathbf{b}$, $\mathbf{c}$ *be points of region* R *in collar* i *of* $\mathrm{EQ}(d,\mathcal{N})$, *which additionally satisfy (3.3.2), that is,*

$$\mathsf{R} = \mathcal{R}\left((\tau_1,\ldots,\tau_{d-1},\vartheta_i),(\upsilon_1,\ldots,\upsilon_{d-1},\vartheta_{i+1})\right)),$$

$$\mathbf{b} = \odot(\beta_1,\beta_2,\ldots,\beta_{d-1},B), \quad \mathbf{c} = \odot(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},B),$$

$$\alpha_k,\beta_k \in [\tau_k,\upsilon_k], k \in \{1,\ldots,d-1\}, \quad B \in [\vartheta_i,\vartheta_{i+1}].$$

We then have $\|\mathbf{c}-\mathbf{b}\| \leqslant \mathsf{w}_i \operatorname{diam} \Pi\,\mathsf{R}$, *where* w_i *is given by (3.3.4).*

Proof. The points $\mathbf{b}$ and $\mathbf{c}$ both have colatitude B. Using the spherical polar coordinates of $\mathbf{b}$ and $\mathbf{c}$ and the mappings $\odot$ and Π we see that if $\Pi\,\mathbf{b} = (b'_1,\ldots,b'_d)$ then

$$\mathbf{b} = (\sin B\ b'_1,\ldots,\sin B\ b'_d,\cos B),$$

and similarly for point $\mathbf{c}$. It follows that $\|\mathbf{c}-\mathbf{b}\| = \sin B\ e(\Pi\,\mathbf{c},\Pi\,\mathbf{b})$.

Since $\Pi\,(b),\Pi\,(c) \in \Pi\,\mathsf{R}$, the Euclidean distance $e(\Pi\,(c),\Pi\,(b))$ is bounded by the diameter of $\Pi\,\mathsf{R}$, so we have $\|\mathbf{c}-\mathbf{b}\| \leqslant \sin B\ \operatorname{diam}\Pi\,\mathsf{R}$. Since $B \in [\vartheta_i,\vartheta_{i+1}]$,

$$\sin B \leqslant \mathsf{w}_i = \max_{\xi\in[\vartheta_i,\vartheta_{i+1}]} \sin\xi.$$

We therefore have $\|\mathbf{c}-\mathbf{b}\| \leqslant \mathsf{w}_i\ \operatorname{diam}\Pi\,\mathsf{R}$. □

We now use these results to prove Lemma 3.3.5.

Proof of Lemma 3.3.5.

Let $\mathbf{a},\mathbf{b}$ be points of region R such that $\|\mathbf{a}-\mathbf{b}\| = \operatorname{diam}\mathsf{R}$ and let

$$\mathbf{a} = \odot(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},A), \quad \mathbf{b} = \odot(\beta_1,\beta_2,\ldots,\beta_{d-1},B),$$

with $\sin B \geqslant \sin A$. Now define $\mathbf{c} := \odot(\alpha_1,\alpha_2,\ldots,\alpha_{d-1},B)$.

By Lemmas 3.11.5, 3.11.6, 2.3.2 and 3.3.4, we then have

$$\begin{aligned}\operatorname{diam}\mathsf{R} = \|\mathbf{a}-\mathbf{b}\| &\leqslant \sqrt{\|\mathbf{a}-\mathbf{c}\|^2+\|\mathbf{c}-\mathbf{b}\|^2}\\ &\leqslant \sqrt{\Upsilon(\delta_i)^2+\mathsf{w}_i^2(\operatorname{diam}\Pi\,\mathsf{R})^2} \leqslant \sqrt{\delta_i^2+\mathsf{w}_i^2(\operatorname{diam}\Pi\,\mathsf{R})^2}.\end{aligned}$$

□

3.11.2 *A continuous model of the partition algorithm*

Rounding the number of collars.

Proof of Lemma 3.5.1.

Assume that $\mathcal{N} > 2$ and $\mathsf{n}_I > \frac{1}{2}$. Therefore, using (3.5.1) and (3.5.4) we have

$$\frac{1}{\rho} = \frac{\mathsf{n}}{\mathsf{n}_I} \in \left(1-\frac{1}{2\mathsf{n}_I}, 1+\frac{1}{2\mathsf{n}_I}\right].$$

Therefore

$$\rho \in \left[\frac{1}{1+\frac{1}{2\mathsf{n}_I}}, \frac{1}{1-\frac{1}{2\mathsf{n}_I}}\right) = \left[\frac{2\mathsf{n}_I}{2\mathsf{n}_I+1}, \frac{2\mathsf{n}_I}{2\mathsf{n}_I-1}\right) = \left[1-\frac{1}{2\mathsf{n}_I+1}, 1+\frac{1}{2\mathsf{n}_I-1}\right).$$

□

Proof of Lemma 3.5.2.

In order,

1. We calculate

$$\nu(2) = \left(\frac{2}{\omega_d}\right)^{\frac{1}{d}}\left(\pi - 2\Theta\left(\frac{\omega_d}{2}\right)\right) = \left(\frac{2}{\omega_d}\right)^{\frac{1}{d}}\left(\pi - 2\frac{\pi}{2}\right) = 0.$$

2. We expand $\nu(\mathcal{N})$ to obtain

$$\nu(\mathcal{N}) = \left(\frac{\mathcal{N}}{\omega_d}\right)^{\frac{1}{d}}\left(\pi - 2\Theta\left(\frac{\omega_d}{\mathcal{N}}\right)\right) = \frac{\pi - 2\vartheta_c}{\mathcal{V}_R} = \mathsf{n}_I.$$

3. We take the derivative of ν and obtain

$$\begin{aligned}
\frac{\partial}{\partial x}\nu(x) &= \pi\frac{\partial}{\partial x}\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}} - 2\pi\frac{\partial}{\partial x}\left(\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}}\Theta\left(\frac{\omega_d}{x}\right)\right) \\
&= \frac{\pi}{d}\omega_d^{-\frac{1}{d}}x^{\frac{1-d}{d}} - 2\frac{\partial}{\partial x}\left(\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}}\Theta\left(\frac{\omega_d}{x}\right)\right) \\
&= \frac{\pi}{d}\omega_d^{-\frac{1}{d}}x^{\frac{1-d}{d}} - 2\Theta\left(\frac{\omega_d}{x}\right)\frac{\partial}{\partial x}\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}} - 2\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}}\frac{\partial}{\partial x}\Theta\left(\frac{\omega_d}{x}\right) \\
&= \frac{\pi}{d}\omega_d^{-\frac{1}{d}}x^{\frac{1-d}{d}} - \frac{2}{d}\omega_d^{-\frac{1}{d}}x^{\frac{1-d}{d}}\Theta\left(\frac{\omega_d}{x}\right) - 2\pi\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}}\frac{\partial}{\partial x}\Theta\left(\frac{\omega_d}{x}\right) \\
&= \left(\pi - 2\Theta\left(\frac{\omega_d}{x}\right)\right)\frac{1}{d}\omega_d^{-\frac{1}{d}}x^{\frac{1-d}{d}} - 2\pi\left(\frac{x}{\omega_d}\right)^{\frac{1}{d}}\frac{\partial}{\partial x}\Theta\left(\frac{\omega_d}{x}\right).
\end{aligned}$$

Using the chain rule and inverse function theorem, we have

$$\frac{\partial}{\partial x}\Theta\left(\frac{\omega_d}{x}\right) = D\Theta\left(\frac{\omega_d}{x}\right)(-\omega_d)x^{-2} = \frac{-\omega_d\ x^{-2}}{D\mathcal{V}\left(\Theta\left(\frac{\omega_d}{x}\right)\right)} = \frac{-\omega_d\ x^{-2}\ d}{\omega_{d-1}\sin\left(\Theta\left(\frac{\omega_d}{x}\right)^d\right)} < 0$$

for $x \geqslant 2$. We also know that $2\Theta(\omega_d/x) \leqslant \pi$ for $x \geqslant 2$. Therefore $D\nu(x) > 0$ and $\nu(x)$ is monotonically increasing in x for $x \geqslant 2$.

□

Rounding the number of regions in a collar.

Proof of Lemma 3.5.3.

To prove (3.5.13), first use (3.2.9) and (3.2.8) to show that for $k \in \{1, \ldots, \mathsf{n}\}$

$$\sum_{i=1}^{k}\mathsf{y}_i = \frac{\mathcal{V}(\vartheta_{F,k+1})}{\mathcal{V}_R} - 1.$$

Then use (3.2.8), (3.2.7) and (3.2.4) to show that

$$\vartheta_{F,\mathsf{n}+1} = \vartheta_c + \mathsf{n}\delta_F = \pi - \vartheta_c.$$

Therefore, by (2.3.44),

$$\frac{\mathcal{V}(\vartheta_{F,\mathsf{n}+1})}{\mathcal{V}_R} = \frac{\omega_d - \mathcal{V}_R}{\mathcal{V}_R} = \mathcal{N} - 1.$$

For (3.5.14), recall from Definition 3.2.10 that $a_0 = 0$ and for $i \in \{1, \dots, \mathsf{n}\}$,

$$\mathsf{a}_i = \sum_{j=1}^{i} (\mathsf{y}_j - \mathsf{m}_j) = \sum_{j=1}^{i-1} (\mathsf{y}_j - \mathsf{m}_j) + \mathsf{y}_i - \mathsf{m}_i = \mathsf{a}_{i-1} + \mathsf{y}_i - \mathsf{m}_i.$$

The result (3.5.14) follows immediately.

For (3.5.15), recalling (3.2.9) we have

$$1 + \sum_{j=1}^{i-1} \mathsf{y}_j = 1 + \frac{\mathcal{V}(\vartheta_{F,i}) - \mathcal{V}(\vartheta_{F,1})}{\mathcal{V}_R} = 1 + \frac{\mathcal{V}(\vartheta_{F,i}) - \mathcal{V}(\vartheta_c)}{\mathcal{V}_R} = \frac{\mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R},$$

where we have used (3.2.8) and (3.2.2). Using Definition 3.2.10 again, we have for $i \in \{1, \dots, \mathsf{n}+1\}$,

$$\frac{\mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R} = 1 + \sum_{j=1}^{i-1} \mathsf{y}_j = 1 + \left(\sum_{j=1}^{i-1} \mathsf{m}_j \right) - \mathsf{a}_{i-1},$$

so

$$\mathcal{V}(\vartheta_{F,i}) = \left(1 + \left(\sum_{j=1}^{i-1} \mathsf{m}_j \right) - \mathsf{a}_{i-1} \right) \mathcal{V}_R = \mathcal{V}(\vartheta_i) - \mathsf{a}_{i-1} \mathcal{V}_R,$$

where we have used (3.2.11). The result (3.5.15) follows.

Finally, by (3.2.9), (3.5.14) and (3.5.15), for $i \in \{1, \dots, \mathsf{n}\}$,

$$\begin{aligned}
\mathcal{V}(\vartheta_{i+1}) - \mathcal{V}(\vartheta_i) &= \mathcal{V}(\vartheta_{F,i+1}) + \mathsf{a}_i \mathcal{V}_R - \mathcal{V}(\vartheta_{F,i}) - \mathsf{a}_{i-1} \mathcal{V}_R \\
&= \mathcal{V}(\vartheta_{F,i+1}) - \mathcal{V}(\vartheta_{F,i}) + (\mathsf{a}_i - \mathsf{a}_{i-1}) \mathcal{V}_R = (\mathsf{y}_i + \mathsf{a}_i - \mathsf{a}_{i-1}) \mathcal{V}_R = \mathsf{m}_i.
\end{aligned}$$

□

Proof of Lemma 3.5.4.

Using Definition 3.2.10, define $z_i := \mathsf{y}_i + \mathsf{a}_{i-1}$.

We first prove (3.5.18). By Definition 3.2.10, we have for $i \in \{1, \dots, \mathsf{n}\}$,

$$\mathsf{a}_i = \mathsf{a}_{i-1} + (\mathsf{y}_i - \mathsf{m}_i) = \mathsf{y}_i + \mathsf{a}_{i-1} - \mathrm{round}(\mathsf{y}_i + \mathsf{a}_{i-1}) = z_i - \mathrm{round}(z_i).$$

Since $\mathsf{y}_i \geqslant \frac{1}{2}$, if $\mathsf{a}_{i-1} \geqslant -\frac{1}{2}$ then $z_i \geqslant 0$, so then $\mathsf{a}_i \in [-\frac{1}{2}, \frac{1}{2})$. But we have $\mathsf{a}_0 = 0$, so the result follows by induction.

We now prove (3.5.17). For $i \in \{1, \ldots, \mathsf{n}\}$, by the analysis above, we have $\mathsf{m}_i = \operatorname{round}(z_i)$ and $z_i \geqslant 0$, so $\mathsf{m}_i \in \mathbb{N}_0$. Also, since $a_0 = 0$, we have $\mathsf{m}_1 > 0$, so $\mathsf{m}_1 \in \mathbb{N}$.

We can now prove (3.5.19). We have, by Definition 3.2.10,

$$\mathsf{a}_n = \sum_{j=1}^{\mathsf{n}} (\mathsf{y}_j - \mathsf{m}_j) = \sum_{j=1}^{\mathsf{n}} \mathsf{y}_j - \sum_{j=1}^{\mathsf{n}} \mathsf{m}_j \in \mathbb{Z}.$$

We know from (3.5.18) that $\mathsf{a}_n \in [-\frac{1}{2}, \frac{1}{2})$, so we must have $\mathsf{a}_n = 0$. Now, by Lemma 3.5.3, we have for $i \in \{1, \ldots, \mathsf{n}\}$,

$$\mathsf{y}_i - \mathsf{m}_i = \mathsf{a}_i - \mathsf{a}_{i-1},$$

so for $i \in \{2, \ldots, \mathsf{n} - 1\}$, by (3.5.18),

$$\mathsf{y}_i - \mathsf{m}_i = \mathsf{a}_i - \mathsf{a}_{i-1}, \in (-1, 1), \text{ and}$$
$$\mathsf{y}_1 - \mathsf{m}_1 = \mathsf{a}_1 - \mathsf{a}_0 = \mathsf{a}_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right),$$

and by (3.5.18),

$$\mathsf{y}_n - \mathsf{m}_n = \mathsf{a}_n - \mathsf{a}_{\mathsf{n}-1} = -\mathsf{a}_{\mathsf{n}-1} \in \left(-\frac{1}{2}, \frac{1}{2}\right],$$

since $a_n = 0$. □

Proof of Lemma 3.5.5.

To prove property 1, suppose that m is symmetric and n is even. Then

$$\mathcal{N} - 2 = \sum_{j=1}^{\mathsf{n}} \mathsf{y}_j = \sum_{j=1}^{\mathsf{n}} \mathsf{m}_j = 2 \sum_{j=1}^{L} \mathsf{m}_j \in 2\mathbb{Z},$$

by Lemma 3.5.4, so $\mathcal{N}$ must be even.

To prove property 2, assume that the sequence m is symmetric. That is, $\mathsf{m}_i = \mathsf{m}_{\mathsf{n}-i+1}$ for $i \in \{1, \ldots, L\}$. Using (3.5.14) we have

$$\mathsf{a}_i = \mathsf{a}_{i-1} + \mathsf{y}_i - \mathsf{m}_i,$$

$$\mathsf{a}_{\mathsf{n}-i+1} = \mathsf{a}_{\mathsf{n}-i} + \mathsf{y}_{\mathsf{n}-i+1} - \mathsf{m}_{\mathsf{n}-i+1},$$

for $i \in \{1, \ldots, \mathsf{n}-1\}$. If we subtract and rearrange, we obtain, for $i \in \{1, \ldots, L\}$

$$\begin{aligned}
\mathsf{a}_i - \mathsf{a}_{\mathsf{n}-i+1} &= \mathsf{a}_{i-1} - \mathsf{a}_{\mathsf{n}-i} + \mathsf{y}_i - \mathsf{y}_{\mathsf{n}-i+1} + \mathsf{m}_{\mathsf{n}-i+1} - \mathsf{m}_i, \\
\mathsf{a}_i - \mathsf{a}_{\mathsf{n}-i+1} &= \mathsf{a}_{i-1} - \mathsf{a}_{\mathsf{n}-i}, \\
\mathsf{a}_i + \mathsf{a}_{\mathsf{n}-i} &= \mathsf{a}_{i-1} + \mathsf{a}_{\mathsf{n}-i+1}.
\end{aligned}$$

Therefore for $i \in \{1, \ldots, L\}$, if $\mathsf{a}_{i-1} + \mathsf{a}_{\mathsf{n}-i+1} = 0$ then $\mathsf{a}_i + \mathsf{a}_{\mathsf{n}-i} = 0$. But by (3.2.10) $\mathsf{a}_0 = 0$, and by (3.5.19) $\mathsf{a}_n = 0$. So, by induction, $\mathsf{a}_i + \mathsf{a}_{\mathsf{n}-i} = 0$ for $i \in \{0, \ldots, L\}$.

To prove property 3, for some $p \in \{0, \ldots, L\}$ assume that the sequence m is symmetric up to m_p. That is, $\mathsf{m}_i = \mathsf{m}_{\mathsf{n}-i+1}$ for $i \in \{1, \ldots, p\}$. By an argument similar to that for property 2, we can show that $\mathsf{a}_p + \mathsf{a}_{\mathsf{n}-p} = 0$. Using (3.5.14) we have

$$\mathsf{a}_{p+1} = \mathsf{a}_p + \mathsf{y}_{p+1} - \mathsf{m}_{p+1}, \quad \mathsf{a}_{\mathsf{n}-p} = \mathsf{a}_{\mathsf{n}-p-1} + \mathsf{y}_{\mathsf{n}-p} - \mathsf{m}_{\mathsf{n}-p}.$$

If we subtract and rearrange, we then obtain

$$\begin{aligned}
\mathsf{a}_{p+1} - \mathsf{a}_{\mathsf{n}-p} &= \mathsf{a}_p - \mathsf{a}_{\mathsf{n}-p-1} + \mathsf{y}_{p+1} - \mathsf{y}_{\mathsf{n}-p} + \mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1}, \\
\mathsf{a}_{p+1} + \mathsf{a}_{\mathsf{n}-p-1} &= \mathsf{a}_p + \mathsf{a}_{\mathsf{n}-p} + \mathsf{y}_{p+1} - \mathsf{y}_{\mathsf{n}-p} + \mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1}, \\
\mathsf{a}_{p+1} + \mathsf{a}_{\mathsf{n}-p-1} &= \mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1},
\end{aligned}$$

since the sequence y is symmetric.

From (3.5.18) we know that $\mathsf{a}_{p+1} + \mathsf{a}_{\mathsf{n}-p-1} \in [-1, 1)$, and from (3.5.17) we know that $\mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1} \in \mathbb{Z}$. There are therefore only two cases:

1. If $\mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1} = \mathsf{a}_{p+1} + \mathsf{a}_{\mathsf{n}-p-1} = -1$, then m is not symmetric, and by (3.5.18),

$$\mathsf{a}_{p+1} = \mathsf{a}_{\mathsf{n}-p-1} = -\frac{1}{2};$$

2. If $\mathsf{m}_{\mathsf{n}-p} - \mathsf{m}_{p+1} = \mathsf{a}_{p+1} + \mathsf{a}_{\mathsf{n}-p-1} = 0$, then m is symmetric up to m_{p+1}.

Property 3 now follows by induction on p. □

Functions which model the regions in a collar.

Proof of Lemma 3.5.6.

In order,

1. Using (3.2.9) we obtain,

$$\mathcal{Y}(\vartheta_{F,i}) = \frac{\mathcal{V}(\vartheta_{F,i} + \delta_F) - \mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R} = \frac{\mathcal{V}(\vartheta_{F,i+1}) - \mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R} = \mathsf{y}_i.$$

2. Using (3.5.15) we have,

$$\mathcal{T}(-\mathsf{a}_{i-1}, \vartheta_{F,i}) = \Theta\big(\mathcal{V}(\vartheta_{F,i}) + a_{i-1}\mathcal{V}_R\big) = \Theta\big(\mathcal{V}(\vartheta_i)\big) = \vartheta_i.$$

3. Again using (3.5.15) we have,

$$\mathcal{B}(\mathsf{a}_i, \vartheta_{F,i}) = \Theta\big(\mathcal{V}(\vartheta_{F,i} + \delta_F) + a_i\mathcal{V}_R\big) = \Theta\big(\mathcal{V}(\vartheta_{F,i+1}) + a_i\mathcal{V}_R\big) = \vartheta_{i+1}.$$

4. Another use of (3.5.15), together with (3.5.16) yields

$$\begin{aligned}\mathcal{M}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathcal{Y}(\vartheta_{F,i}) - \mathsf{a}_{i-1} + \mathsf{a}_i &= \frac{\mathcal{V}(\vartheta_{F,i+1}) - \mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R} - \mathsf{a}_{i-1} + \mathsf{a}_i \\ &= \frac{\mathcal{V}(\vartheta_{F,i+1})}{\mathcal{V}_R} + \mathsf{a}_i - \frac{\mathcal{V}(\vartheta_{F,i})}{\mathcal{V}_R} - \mathsf{a}_{i-1} = \frac{\mathcal{V}(\vartheta_{i+1}) - \mathcal{V}(\vartheta_i)}{\mathcal{V}_R} = \mathsf{m}_i.\end{aligned}$$

5. Using (3.5.29), (3.5.28) and (3.3.3), we have

$$\Delta(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathcal{B}(\mathsf{a}_i, \vartheta_{F,i}) - \mathcal{T}(-\mathsf{a}_{i-1}, \vartheta_{F,i}) = \vartheta_{i+1} - \vartheta_i = \delta_i.$$

6. Using (3.5.29), (3.5.28) and (3.3.4) we have

$$\mathcal{W}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \max_{\xi \in [\mathcal{T}(-\mathsf{a}_{i-1}, \vartheta_{F,i}), \mathcal{B}(\mathsf{a}_i, \vartheta_{F,i})]} \sin \xi = \max_{\xi \in [\vartheta_i, \vartheta_{i+1}]} \sin \xi = \mathsf{w}_i.$$

7. Finally, using (3.5.30) and (3.5.32) we have

$$\mathcal{P}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \mathcal{W}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i})\, \mathcal{M}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i})^{\frac{1}{1-d}} = \mathsf{w}_i \mathsf{m}_i^{\frac{1}{1-d}} = \mathsf{p}_i.$$

□

Symmetries of the continuous analogs.

Proof of Lemma 3.5.7.

Using (2.3.44), we have

$$\mathcal{Y}(\pi - \vartheta) = \frac{\mathcal{V}(\pi - \vartheta + \delta_F) - \mathcal{V}(\pi - \vartheta)}{\mathcal{V}_R} = \frac{\omega_d - \mathcal{V}(\vartheta - \delta_F) - \omega_d + \mathcal{V}(\vartheta)}{\mathcal{V}_R} = \mathcal{Y}(\vartheta - \delta_F).$$

□

Proof of Lemma 3.5.8.

Using (2.3.46), we have

$$\begin{aligned} \mathcal{T}(\tau, \pi - \vartheta) &= \Theta\left(\mathcal{V}(\pi - \vartheta) - \tau \mathcal{V}_R\right) = \Theta\left(\omega_d - \mathcal{V}(\vartheta) - \tau \mathcal{V}_R\right) \\ &= \pi - \Theta\left(\mathcal{V}(\vartheta) + \tau \mathcal{V}_R\right) = \pi - \mathcal{B}(\tau, \vartheta - \delta_F), \end{aligned}$$

and so

$$\mathcal{B}(\beta, \pi - \vartheta) = \mathcal{B}\left(\beta, (\pi - \vartheta + \delta_F) - \delta_F\right) = \pi - \mathcal{T}\left(\beta, \pi - (\pi - \vartheta + \delta_F)\right) = \pi - \mathcal{T}(\beta, \vartheta - \delta_F).$$

□

Proof of Lemma 3.5.9.

For $\mathcal{M}$, we have

$$\mathcal{M}(\tau, \beta, \pi - \vartheta) = \mathcal{Y}(\pi - \vartheta) + \tau + \beta = \mathcal{Y}(\vartheta - \delta_F) + \beta + \tau = \mathcal{M}(\beta, \tau, \vartheta - \delta_F).$$

For Δ, we have

$$\begin{aligned}\Delta(\tau, \beta, \pi - \vartheta) &= \mathcal{B}(\beta, \pi - \vartheta) - \mathcal{T}(\tau, \pi - \vartheta) = \pi - \mathcal{T}(\beta, \vartheta - \delta_F) - \pi + \mathcal{B}(\tau, \vartheta - \delta_F) \\ &= \Delta(\beta, \tau, \vartheta - \delta_F).\end{aligned}$$

For $\mathcal{W}$, first note that

$$\sin \mathcal{T}(\tau, \pi - \vartheta) = \sin\left(\pi - \mathcal{B}(\tau, \vartheta - \delta_F)\right) = \sin \mathcal{B}(\tau, \vartheta - \delta_F)$$

and similarly

$$\sin \mathcal{B}(\beta, \pi - \vartheta) = \sin \mathcal{T}(\beta, \vartheta - \delta_F).$$

Now recall from (3.5.25) that

$$\mathcal{W}(\tau, \beta, \vartheta) = \max_{\xi \in [\mathcal{T}(\tau, \vartheta), \mathcal{B}(\beta, \vartheta)]} \sin \xi,$$

so

$$\begin{aligned}\mathcal{W}(\tau, \beta, \pi - \vartheta) &= \max_{\xi \in [\mathcal{T}(\tau, \pi - \vartheta), \mathcal{B}(\beta, \pi - \vartheta)]} \sin \xi, = \max_{\xi \in [\pi - \mathcal{B}(\tau, \vartheta - \delta_F), \pi - \mathcal{T}(\beta, \vartheta - \delta_F)]} \sin \xi \\ &= \max_{\xi \in [\mathcal{T}(\beta, \vartheta - \delta_F), \mathcal{B}(\tau, \vartheta - \delta_F)]} \sin(\pi - \xi) = \max_{\xi \in [\mathcal{T}(\beta, \vartheta - \delta_F), \mathcal{B}(\tau, \vartheta - \delta_F)]} \sin \xi \\ &= \mathcal{W}(\beta, \tau, \vartheta - \delta_F).\end{aligned}$$

For $\mathcal{P}$, we have

$$\mathcal{P}(\tau,\beta,\pi-\vartheta) = \mathcal{W}(\tau,\beta,\pi-\vartheta)\,\mathcal{M}(\tau,\beta,\pi-\vartheta)^{\frac{1}{1-d}} = \mathcal{W}(\beta,\tau,\vartheta-\delta_F)\,\mathcal{M}(\beta,\tau,\vartheta-\delta_F)^{\frac{1}{1-d}}$$
$$= \mathcal{P}(\beta,\tau,\vartheta-\delta_F).$$

□

Feasible domains.

Proof of Lemma 3.5.11.

Assume that $d > 1$ and $\mathcal{N} > 2$ and that $\mathrm{EQ}(d-1)$ has diameter bound κ. From Corollary 3.3.3 we see that we need only consider the regions of $\mathrm{EQ}(d,\mathcal{N})$ which are contained in collars.

For $\mathbb{D}$ as per Definition 3.5.10, if $i \in \{1,\ldots,\mathsf{n}\}$, where n is the number of collars in $\mathrm{EQ}(d,\mathcal{N})$, Lemma 3.5.4 ensures that $(-\mathsf{a}_{i-1},\mathsf{a}_i,\vartheta_{F,i}) \in \mathbb{D}$.

We therefore have the inequality

$$\max_{i\in\{1,\ldots,\mathsf{n}\}} f(-\mathsf{a}_{i-1},\mathsf{a}_i,\vartheta_{F,i}) \leqslant \max_{(\tau,\beta,\vartheta)\in\mathbb{D}} f(\tau,\beta,\vartheta) = \max_{\mathbb{D}} f$$

for any f defined on $\mathbb{D}$. Lemma 3.5.6 and (3.4.2) then imply the current result. □

Symmetries of the feasible domain $\mathbb{D}$*.*

Proof of Lemma 3.5.12.

Assume that $(\tau,\beta,\vartheta) \in \mathbb{D}_-$. From Lemma 3.5.9 we know that

$$f(\tau,\beta,\vartheta) = f(\beta,\tau,\pi-\vartheta-\delta_F),$$

so we have $f(\tau',\beta',\vartheta') = f(\tau,\beta,\vartheta)$ where

$$\tau' := \beta, \beta' := \tau \quad \text{and} \quad \vartheta' := \pi-\vartheta-\delta_F.$$

For $\mathbb{D}_{m-}$, we have $\vartheta \in \left[\frac{\pi}{2} - \frac{\delta_F}{2}, \pi - \vartheta_c - 2\delta_F\right]$. Then $(\tau', \beta', \vartheta') \in \mathbb{D}_{m+}$, since $\pi - \vartheta - \delta_F \in \left[\vartheta_{F,2}, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$.

For $\mathbb{D}_b$, we have $\vartheta = \pi - \vartheta_c - \delta_F$. Then $(\tau', \beta', \vartheta') \in \mathbb{D}_t$, since $\pi - \vartheta - \delta_F = \vartheta_c$. $\square$

Estimates.

Proof of Lemma 3.5.14.

For $x \geqslant 2$ let

$$f(x) := x^{\frac{1}{d}} \sin \Theta\left(\frac{\omega_d}{x}\right).$$

Then $f(\mathcal{N}) = \mathcal{N}^{\frac{1}{d}} \sin \vartheta_c$. Since $\Theta(v) > 0$ for $v > 0$ we see that $f(x) > 0$.

We now compute the derivative

$$\begin{aligned}
\frac{\partial}{\partial x} f(x)^d &= \sin^d \Theta\left(\frac{\omega_d}{x}\right) + x \frac{\partial}{\partial x} \sin^d \Theta\left(\frac{\omega_d}{x}\right) \\
&= \sin^d \Theta\left(\frac{\omega_d}{x}\right) + x\ d\ \sin^{d-1} \Theta\left(\frac{\omega_d}{x}\right) \cos \Theta\left(\frac{\omega_d}{x}\right) \frac{\partial}{\partial x} \Theta\left(\frac{\omega_d}{x}\right).
\end{aligned}$$

Using the inverse function theorem, we have

$$\frac{\partial}{\partial x} \Theta\left(\frac{\omega_d}{x}\right) = \frac{1}{D\mathcal{V}\left(\Theta\left(\frac{\omega_d}{x}\right)\right)} \frac{\partial}{\partial x} \left(\frac{\omega_d}{x}\right) = \frac{-\omega_d}{x^2 \omega_{d-1} \sin^{d-1} \Theta\left(\frac{\omega_d}{x}\right)},$$

so

$$\frac{\partial}{\partial x} f(x)^d = \sin^d \Theta\left(\frac{\omega_d}{x}\right) - \frac{\omega_d}{\omega_{d-1}} \frac{d}{x} \cos \Theta\left(\frac{\omega_d}{x}\right).$$

From (2.3.50), for $x > 2$ we have

$$0 < \frac{\omega_d}{x} < \frac{\omega_{d-1}}{d} \frac{\sin^d \Theta\left(\frac{\omega_d}{x}\right)}{\cos \Theta\left(\frac{\omega_d}{x}\right)},$$

so

$$0 < \frac{\omega_d}{\omega_{d-1}} \frac{d}{x} \cos \Theta\left(\frac{\omega_d}{x}\right) < \sin^d \Theta\left(\frac{\omega_d}{x}\right)$$

and therefore

$$\frac{\partial}{\partial x} f(x)^d > 0,$$

which implies that $f(x)$ is monotonically increasing in x for $x > 2$.

On the other hand (2.2.6) and (3.5.46) result in the estimate

$$f(x) = x^{\frac{1}{d}} \ \sin \Theta \left(\frac{\omega_d}{x} \right) \in [J_c(y), J_c(y)^{\frac{1-d}{d}}] \ \left(\frac{\omega_d \ d}{\omega_{d-1}} \right)^{\frac{1}{d}}$$

for $x > y \geqslant 2$, with $J_c(y) \nearrow 1$ as $y \to \infty$. Therefore

$$x^{\frac{1}{d}} \ \sin \Theta \left(\frac{\omega_d}{x} \right) \nearrow \left(\frac{\omega_d \ d}{\omega_{d-1}} \right)^{\frac{1}{d}}$$

as $x \to \infty$. The results (3.5.47) and (3.5.48) follow. □

Proof of Lemma 3.5.15.

For $\mathcal{N} > 2$, the EQ algorithm at (3.2.5) defines

$$\mathsf{n} = \max\left(1, \mathrm{round}\left(\mathsf{n}_I\right)\right),$$

For $\mathcal{N} > \mathcal{N}_0(1/2)$, Lemma 3.5.2 implies that $\mathcal{N}_0(1/2) > 2$ and (3.5.9) gives us $\mathsf{n}_I > \frac{1}{2}$. Together with (3.2.5), this implies that

$$\mathsf{n} \leqslant \mathsf{n}_I + \frac{1}{2} \tag{3.11.1}$$

for $\mathcal{N} > \mathcal{N}_0(1/2)$.

From (3.2.4) we have

$$\mathsf{n}_I = \frac{\pi - 2\vartheta_c}{\delta_I}.$$

From the estimate (3.5.46) we have

$$\vartheta_c \geqslant \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \delta_I.$$

From (2.3.56) we have

$$\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \geqslant \pi^{-\frac{1}{2}},$$

and so

$$\vartheta_c \geqslant \frac{\delta_I}{\sqrt{\pi}} > \frac{\delta_I}{2}.$$

Therefore (3.11.1) implies that

$$\mathsf{n} \leqslant \mathsf{n}_I + \frac{1}{2} \leqslant \mathsf{n}_I + 1 = \frac{\pi - 2\vartheta_c + \delta_I}{\delta_I} \leqslant \frac{\pi}{\delta_I}.$$

$\square$

Proof of Lemma 3.5.16.

If $\iota \leqslant \frac{13}{4}$ then

$$\frac{10}{9}(\iota - 1) \leqslant \frac{5}{2}. \tag{3.11.2}$$

By Lemma 3.5.2, if $x \geqslant \mathcal{N}_0(5)$ then $\nu(x) \geqslant 5$ so that

$$\frac{\pi}{2} - \Theta\left(\frac{\omega_d}{x}\right) \geqslant \frac{5}{2} \left(\frac{\omega_d}{x}\right)^{\frac{1}{d}}. \tag{3.11.3}$$

Also, by (3.5.11) we have $\rho_H(x) \leqslant \frac{10}{9}$. Therefore from (3.11.2) and (3.11.3) we have

$$\rho_H(x)(\iota - 1)\left(\frac{\omega_d}{x}\right)^{\frac{1}{d}} \leqslant \frac{5}{2} \left(\frac{\omega_d}{x}\right)^{\frac{1}{d}} \leqslant \frac{\pi}{2} - \Theta\left(\frac{\omega_d}{x}\right).$$

□

Proof of Lemma 3.5.17.

The result follows from (2.3.57), since both $J_{F,(1+\eta)}(x) \nearrow 1$ and $\rho_L(x) \nearrow 1$ as $x \to \infty$. □

Proof of Lemma 3.5.18.

The result follows from (2.3.58) and Lemma 3.5.17, since both $J_{F,2}(x) \nearrow 1$ and $\rho_L(x) \nearrow 1$ as $x \to \infty$. □

Proof of Lemma 3.5.19.

We have $x \geqslant \mathcal{N}_0(5)$ and $1+\eta < \frac{13}{4}$. By Lemma 3.5.16 the condition (3.5.55) holds for $\iota = 1+\eta$ and we can therefore apply the lower estimate of (3.5.58).

By (2.3.47), (3.5.46), (3.5.58) and (3.5.62) we now have

$$\mathcal{V}(\vartheta_c + \eta\delta_F) = \mathcal{V}(\vartheta_{F,(1+\eta)}) \geqslant J_{F,(1+\eta)}(x)^{d-1} \left(1 + \eta\ \rho_L(x)\ \left(\frac{\omega_{d-1}}{d}\right)^{\frac{1}{d}}\right)^d \mathcal{V}_R > \frac{3}{2}\mathcal{V}_R.$$

□

Proof of Corollary 3.5.20.

For $\vartheta \in \left[\vartheta_c, \frac{\pi}{2} - \eta\frac{\delta_F}{2}\right]$, the result follows immediately from (3.5.65) and (3.5.66).

Otherwise, let $\bar{\vartheta} := \pi - \eta\delta_F - \vartheta$. Then for $\vartheta \in \left(\frac{\pi}{2} - \eta\frac{\delta_F}{2}, \pi - \vartheta_c - \eta\delta_F\right]$ we have $\bar{\vartheta} \in \left[\vartheta_c, \frac{\pi}{2} - \eta\frac{\delta_F}{2}\right]$. Using Lemma 2.3.16 we then have

$$\begin{aligned}\mathcal{V}(\vartheta + \eta\delta_F) - \mathcal{V}(\vartheta) &= \mathcal{V}(\pi - \bar{\vartheta}) - \mathcal{V}(\pi - \eta\delta_F - \bar{\vartheta}) = \omega_d - \mathcal{V}(\bar{\vartheta}) - \omega_d + \mathcal{V}(\bar{\vartheta} + \eta\delta_F) \\ &= \mathcal{V}(\bar{\vartheta} + \eta\delta_F) - \mathcal{V}(\bar{\vartheta}) > \frac{\mathcal{V}_R}{2}.\end{aligned}$$

□

Proof of Lemma 3.5.23.

We have $\mathsf{m}_0 = 1$. For $\mathcal{N} = 2$, $\mathsf{m}_1 = 1$, the single region of the south polar cap. For $\mathcal{N} \geqslant 1$ we have $\mathcal{N}^{\frac{d-1}{d}} \geqslant 1$.

For $\mathcal{N} \in [3, \mathcal{N}_0(1/2))$, using (3.2.5), Lemma 3.5.2 and (3.5.8) we see that we have one collar containing $\mathcal{N} - 2$ regions. For $\mathcal{N} \in (2, \mathcal{N}_0(1/2))$ we have

$$\left(\mathcal{N}_0(1/2)^{\frac{1}{d}} - 2\ \mathcal{N}_0(1/2)^{\frac{1-d}{d}}\right)\ \mathcal{N}^{\frac{d-1}{d}} \geqslant \left(\mathcal{N}^{\frac{1}{d}} - 2\ \mathcal{N}^{\frac{1-d}{d}}\right)\ \mathcal{N}^{\frac{d-1}{d}} = \mathcal{N} - 2.$$

For $\mathcal{N} \geqslant x \geqslant \mathcal{N}_0(1/2)$ we use (3.5.10), (3.5.11), Definition 3.5.10, (3.5.23), (3.5.30) and (3.5.73) to show that for $i \in \{1, \ldots, \mathsf{n}\}$ we have

$$\begin{aligned}
\mathsf{m}_i &\leqslant \rho\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}}\ \mathcal{N}^{\frac{d-1}{d}} + 1 \\
&\leqslant \rho_H(x)\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}}\ \mathcal{N}^{\frac{d-1}{d}} + 1.
\end{aligned}$$

From (3.5.12) we know that $\rho_H(x) \leqslant \rho_H\big(\mathcal{N}_0(1/2)\big)$ for $x \geqslant \mathcal{N}_0(1/2)$. We therefore have

$$\rho_H(x)\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}}\ \mathcal{N}^{\frac{d-1}{d}} + 1 \leqslant \rho_H\big(\mathcal{N}_0(1/2)\big)\ \omega_{d-1}\ \omega_d^{\frac{1-d}{d}}\ \mathcal{N}^{\frac{d-1}{d}} + \mathcal{N}^{\frac{d-1}{d}}.$$

□

Bounds.

Proof of Lemma 3.5.24.

Assume that $d > 1$ and $\mathcal{N} > 1$. From Lemma 3.3.1, we know that the diameter of each of the polar caps of the partition $\mathrm{EQ}(d, \mathcal{N})$ is $2 \sin \vartheta_c$, where ϑ_c is defined by (3.2.2). From (3.5.48) we have the estimate

$$2 \sin \vartheta_c \leqslant 2 \left(\frac{\omega_d\ d}{\omega_{d-1}}\right)^{\frac{1}{d}} \mathcal{N}^{-\frac{1}{d}}.$$

for $\mathcal{N} \geqslant x \geqslant 2$. □

Proof of Lemma 3.5.25.

Throughout this proof, we assume that $\mathcal{N} > x$ where $x \geqslant \mathcal{N}_0(5)$, with $\mathcal{N}_0$ defined by (3.5.8), so that $\mathsf{n} \geqslant 5$. Using Corollary 3.5.13, we also assume that $(\tau, \beta, \vartheta) \in \mathbb{D}_+$.

For the top collar, $(\tau,\beta,\vartheta)\in\mathbb{D}_t$, (3.5.38) gives $\tau=0$, $\beta\in\left[-\frac{1}{2},\frac{1}{2}\right]$, $\vartheta=\vartheta_c$. Using Lemma 2.3.16 we have

$$\mathcal{V}\left(\mathcal{B}(\beta,\vartheta_c)\right)=\mathcal{V}(\vartheta_c+\delta_F)+\beta\mathcal{V}_R\leqslant\mathcal{V}(\vartheta_c+\delta_F)+\frac{\mathcal{V}_R}{2}.$$

Since $\mathsf{n}\geqslant 5$, we have $\vartheta_c+\delta_F\in[\vartheta_c,\pi-\vartheta_c-\eta\delta_F]$, and we can use (3.5.67) to obtain

$$\mathcal{V}\left(\mathcal{B}(\beta,\vartheta_c)\right)\leqslant\mathcal{V}(\vartheta_c+\delta_F)+\frac{\mathcal{V}_R}{2}<\mathcal{V}\big(\vartheta_c+(1+\eta)\delta_F\big).$$

Therefore, using Lemma 2.3.16 again, we have

$$\mathcal{B}(\beta,\vartheta_c)<\vartheta_c+(1+\eta)\delta_F. \tag{3.11.4}$$

Therefore (3.5.21) and (3.5.24) yield

$$\Delta(\tau,\beta,\vartheta)=\Delta(0,\beta,\vartheta_c)=\mathcal{B}(\beta,\vartheta_c)-\mathcal{T}(0,\vartheta_c)=\mathcal{B}(\beta,\vartheta_c)-\vartheta_c<(1+\eta)\delta_F.$$

For $(\tau,\beta,\vartheta)\in\mathbb{D}_{m+}$ (3.5.44) gives $\tau\in\left[-\frac{1}{2},\frac{1}{2}\right]$, $\beta\in\left[-\frac{1}{2},\frac{1}{2}\right]$, $\vartheta\in\left[\vartheta_{F,2},\frac{\pi}{2}-\frac{\delta_F}{2}\right]$. Since $\mathsf{n}\geqslant 5$, we have $\vartheta+\delta_F\in[\vartheta_c,\pi-\vartheta_c-\eta\delta_F]$, since

$$\vartheta_c+\frac{3}{2}\delta_F<\vartheta_c+2\delta_F<\frac{\pi}{2},$$

yielding

$$\vartheta+\delta_F\leqslant\frac{\pi}{2}+\frac{\delta_F}{2}<\pi-\vartheta_c-\delta_F.$$

Using Lemma 2.3.16, (3.5.22) and (3.5.67) we now have

$$\mathcal{V}\big(\mathcal{B}(\beta,\vartheta)\big)=\mathcal{V}(\vartheta+\delta_F)+\beta\mathcal{V}_R\leqslant\mathcal{V}(\vartheta+\delta_F)+\frac{\mathcal{V}_R}{2}<\mathcal{V}\big(\vartheta+(1+\eta)\delta_F\big).$$

Using Lemma 2.3.16 again, we therefore have

$$\mathcal{B}(\beta, \vartheta) < \vartheta + (1 + \eta)\delta_F. \tag{3.11.5}$$

Since $\vartheta - \eta\delta_F > \vartheta_c$, using Lemma 2.3.16, (3.5.21) and (3.5.67) we also have

$$\mathcal{V}\big(\mathcal{T}(\tau, \vartheta)\big) = \mathcal{V}(\vartheta) + \tau\mathcal{V}_R \geqslant \mathcal{V}(\vartheta) - \frac{\mathcal{V}_R}{2} > \mathcal{V}(\vartheta - \eta\delta_F),$$

so that

$$\vartheta - \eta\delta_F < \mathcal{T}(\tau, \vartheta). \tag{3.11.6}$$

Combining (3.11.5) and (3.11.6), and using (3.5.24) we therefore have

$$\Delta(\tau, \beta, \vartheta) = \mathcal{B}(\beta, \vartheta) - \mathcal{T}(\tau, \vartheta) < (1 + 2\eta)\delta_F.$$

The estimate (3.5.52) now yields

$$\Delta(\tau, \beta, \vartheta) < K_\Delta(x)\mathcal{N}^{-\frac{1}{d}},$$

where

$$K_\Delta(x) := (1 + 2\eta)\ \rho_H(x)\ \omega_d^{\frac{1}{d}}, \tag{3.11.7}$$

with $\rho_H(x)$ defined by (3.5.11).

We also have

$$K_\Delta(x) \searrow K_\Delta(\infty) := (1 + 2\eta)\ \omega_d^{\frac{1}{d}} \tag{3.11.8}$$

as $x \to \infty$, since $\rho_H(x) \searrow 1$ as $x \to \infty$, by (3.5.12). $\square$

Proof of Lemma 3.5.26.

Throughout this proof, we assume that $\mathcal{N} > x$ where $x \geqslant \mathcal{N}_0(5)$, with $\mathcal{N}_0$ defined by (3.5.8), and x satisfies (3.5.62) and (3.5.63). Lemma 3.5.18 assures us that such an x exists. As a consequence, we have at least 5 collars,

$$\mathsf{n}_I \geqslant 5, \quad \mathsf{n} \geqslant 5, \tag{3.11.9}$$

and (3.5.69) and (3.5.71) are satisfied. We also use Corollary 3.5.13, and assume that $(\tau, \beta, \vartheta) \in \mathbb{D}_+$. The reasons for assuming that x satisfies (3.5.62) and (3.5.63) will become clear in the course of the proof.

Within this proof, we use the abbreviation

$$\gamma := 1 + \eta, \tag{3.11.10}$$

where η is defined by (3.5.61).

We now divide $\mathbb{D}_+$ into $\mathbb{D}_t$ and $\mathbb{D}_{m+}$, as per (3.5.38) and (3.5.44), and we examine $\mathbb{D}_t$ first.

Bounds for $\mathcal{P}$ in $\mathbb{D}_t$.

For the top collar, $(\tau, \beta, \vartheta) \in \mathbb{D}_t$, (3.5.38) gives $\tau = 0$, $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\vartheta = \vartheta_c$. Therefore, in $\mathbb{D}_t$, we consider

$$\widetilde{P}(\beta) := \mathcal{P}(0, \beta, \vartheta_c) = \widetilde{W}(\beta)\widetilde{M}(\beta)^{\frac{1}{1-d}},$$

where

$$\widetilde{W}(\beta) := \mathcal{W}(0, \beta, \vartheta_c) = \max_{\xi \in [\mathcal{T}(0,\vartheta_c), \mathcal{B}(\beta,\vartheta_c)]} \sin \xi,$$
$$\widetilde{M}(\beta) := \mathcal{M}(0, \beta, \vartheta_c) = \frac{\mathcal{V}(\vartheta_{F,2}) - \mathcal{V}(\vartheta_c)}{\mathcal{V}_R} + \beta.$$

We made weaker assumptions for $\mathcal{N}$ and x for the analysis of $\mathbb{D}_t$ in the proof of Lemma 3.5.25, in particular, we assumed there as we do here that x is large enough that (3.11.9) holds. We therefore see that (3.11.4) holds here, so that the top collar is completely contained in the Northern hemisphere, in other words, $\mathcal{B}(\beta, \vartheta_c) < \frac{\pi}{2}$. We therefore have

$$\widetilde{W}(\beta) = \sin \mathcal{B}(\beta, \vartheta_c).$$

As a consequence,

$$\begin{aligned}
\widetilde{P}(\beta) &= \sin \mathcal{B}(\beta, \vartheta_c) \left(\frac{\mathcal{V}(\vartheta_{F,2})}{\mathcal{V}_R} - 1 + \beta \right)^{\frac{1}{1-d}} \\
&= \sin \Theta\big(\mathcal{V}(\vartheta_{F,2}) + \beta \mathcal{V}_R\big) \left(\frac{\mathcal{V}(\vartheta_{F,2})}{\mathcal{V}_R} - 1 + \beta \right)^{\frac{1}{1-d}} \\
&= \sin \Theta\big((s+\beta)\mathcal{V}_R\big)\, (s + \beta - 1)^{\frac{1}{1-d}},
\end{aligned}$$

where

$$s := \frac{\mathcal{V}(\vartheta_{F,2})}{\mathcal{V}_R}. \tag{3.11.11}$$

By (3.5.38) and (3.5.70) we have $\beta \geqslant -\frac{1}{2}$ and $s > 9/2$. Therefore $s + \beta - 1 > 3$. Also,

$$\Theta\big((s+\beta)\mathcal{V}_R\big) = \mathcal{B}(\beta, \vartheta_c) < \frac{\pi}{2}.$$

Applying (2.2.6), (2.3.48) and (3.5.56) we now have $0 < \widetilde{P}(\beta) \leqslant \widetilde{Q}(\beta)$, where

$$\widetilde{Q}(\beta) := r(x)(s+\beta)^{\frac{1}{d}} (s + \beta - 1)^{\frac{1}{1-d}}\ \delta_I \tag{3.11.12}$$

and

$$r(x) := J_{F,(2+\eta)}(x)^{\frac{1-d}{d}} \left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}}. \tag{3.11.13}$$

This estimate requires that

$$\Theta\left(\frac{\omega_d}{x}\right) + \gamma\ \rho_H(x)\left(\frac{\omega_d}{x}\right)^{\frac{1}{d}} \leqslant \frac{\pi}{2}. \tag{3.11.14}$$

Since $\rho_H(x) \searrow 1$ as $x \to \infty$, the expression on the left of (3.11.14) is positive and monotonic decreasing in x, and so this condition holds for x sufficiently large.

Since $J_{F,(2+\eta)}(x) \nearrow 1$ as $x \to \infty$, we see that

$$r(x) \searrow \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \tag{3.11.15}$$

as $x \to \infty$.

To determine whether $\widetilde{Q}(\beta)$ is monotonic in β, we compute

$$\begin{aligned}
\frac{\partial}{\partial\beta}\widetilde{Q}(\beta) &= r(x)\left(\frac{1}{d}(s+\beta)^{\frac{1-d}{d}}(s+\beta-1)^{\frac{1}{1-d}} - \frac{1}{d-1}(s+\beta)^{\frac{1}{d}}(s+\beta-1)^{\frac{d}{1-d}}\right)\ \delta_I \\
&= \widetilde{Q}(\beta)\left(\frac{1}{d(s+\beta)} - \frac{1}{(d-1)(s+\beta-1)}\right) \\
&= \widetilde{Q}(\beta)\frac{-d-s-\beta+1}{d\ (d-1)(s+\beta)(s+\beta-1)} < 0,
\end{aligned}$$

since $d \geqslant 2,\ s > 9/2,\ \beta \geqslant -\frac{1}{2}$. The maximum of $\widetilde{Q}(\beta)$ therefore occurs when $\beta = -\frac{1}{2}$. Therefore

$$\widetilde{P}(\beta) \leqslant \widetilde{Q}\left(-\frac{1}{2}\right). \tag{3.11.16}$$

Estimate for $\mathbb{D}_t$.

From (3.5.58) and (3.11.11) we see that $s \in [s_{L,2}, s_{H,2}]$ and therefore (3.11.16) yields

$$\widetilde{P}(\beta) \leqslant r(x)\left(s_{H,2}(x) - \frac{1}{2}\right)^{\frac{1}{d}}\left(s_{L,2}(x) - \frac{3}{2}\right)^{\frac{1}{1-d}}\ \delta_I.$$

Using (3.5.60) and (3.11.15) we see that

$$r(x)\left(s_{H,2}(x)-\frac{1}{2}\right)^{\frac{1}{d}}\left(s_{L,2}(x)-\frac{3}{2}\right)^{\frac{1}{1-d}} \searrow \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}}\left(s_2-\frac{1}{2}\right)^{\frac{1}{d}}\left(s_2-\frac{3}{2}\right)^{\frac{1}{1-d}}$$

as $x \to \infty$.

Assuming that $\mathcal{N} > x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and that x satisfies (3.5.62), (3.5.63) and (3.11.14), the resulting estimate is then

$$\widetilde{P}(\beta) \leqslant \mathsf{C}_t(x)\mathcal{N}^{-\frac{1}{d}},$$

where

$$\mathsf{C}_t(x) := r(x)\left(s_{H,2}(x)-\frac{1}{2}\right)^{\frac{1}{d}}\left(s_{L,2}(x)-\frac{3}{2}\right)^{\frac{1}{1-d}}\omega_d^{\frac{1}{d}}, \tag{3.11.17}$$

with

$$\mathsf{C}_t(x) \searrow \mathsf{C}_t(\infty) := \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}}\left(s_2-\frac{1}{2}\right)^{\frac{1}{d}}\left(s_2-\frac{3}{2}\right)^{\frac{1}{1-d}}\omega_d^{\frac{1}{d}} \tag{3.11.18}$$

as $x \to \infty$.

Bounds for $\mathcal{P}$ in $\mathbb{D}_{m+}$.

In the following arguments the notation suppresses the explicit dependence of various functions on τ and β, wherever sensible.

By (3.5.22), (3.5.68) and (3.5.67), for $\vartheta \in \left[\vartheta_{F,2}, \frac{\pi}{2}-\frac{\delta_F}{2}\right]$ we have

$$\begin{aligned}\mathcal{V}\left(\mathcal{B}(\beta,\vartheta)\right) &= \mathcal{V}(\vartheta+\delta_F)+\beta\mathcal{V}_R \leqslant \mathcal{V}(\vartheta+\delta_F)+\frac{\mathcal{V}_R}{2}\\ &< \mathcal{V}\big(\vartheta+(1+\eta)\delta_F\big) = \mathcal{V}(\vartheta+\gamma\delta_F).\end{aligned}$$

Therefore by Lemma 2.3.16, we have $\mathcal{B}(\beta,\vartheta) < \vartheta+\gamma\delta_F$.

If $\vartheta \in [\vartheta_{F,2}, \frac{\pi}{2} - \gamma\delta_F]$, then for $\xi \in [\mathcal{T}(\tau,\vartheta), \mathcal{B}(\beta,\vartheta)]$, we have

$$\sin\xi \leqslant \sin(\vartheta + \gamma\delta_F),$$

and therefore $\mathcal{W}(\vartheta) \leqslant \sin(\vartheta + \gamma\delta_F)$.

Motivated by (3.11.25), define

$$\mathbb{D}_{mL} = \{(\tau,\beta,\vartheta) \in \mathbb{D}_{m+} \mid \vartheta \leqslant \frac{\pi}{2} - \gamma\delta_F\}, \tag{3.11.19}$$

$$\mathbb{D}_{mH} = \{(\tau,\beta,\vartheta) \in \mathbb{D}_{m+} \mid \vartheta > \frac{\pi}{2} - \gamma\delta_F\}, \tag{3.11.20}$$

Since we have $\mathsf{n} \geqslant 5$, we have

$$\vartheta_{F,2} = \vartheta_c + \delta_F \leqslant \frac{\pi}{2} - \frac{3}{2}\delta_F < \frac{\pi}{2} - \gamma\delta_F < \frac{\pi}{2} - \frac{\delta_F}{2}.$$

Therefore both $\mathbb{D}_{mL}$ and $\mathbb{D}_{mH}$ are non-empty.

The domain $\mathbb{D}_{mL}$.

For $\mathbb{D}_{mL}$ we have $\vartheta \in [\vartheta_{F,2}, \frac{\pi}{2} - \gamma\delta_F]$. Define

$$Q(\vartheta) := \sin(\vartheta + \gamma\delta_F)\,\mathcal{M}(\vartheta)^{\frac{1}{1-d}}, \tag{3.11.21}$$

We then have, for $\vartheta \in [\vartheta_{F,2}, \frac{\pi}{2} - \gamma\delta_F]$,

$$\mathcal{P}(\vartheta) \leqslant Q(\vartheta). \tag{3.11.22}$$

From (2.3.42) we know that for $\vartheta \in [0, \frac{\pi}{2} - \delta_F]$,

$$\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta) \in \left[D\mathcal{V}(\vartheta), D\mathcal{V}(\vartheta + \delta_F)\right]\ \delta_F, \tag{3.11.23}$$

and therefore, from (3.5.20) and (3.5.23)

$$\mathcal{Y}(\vartheta) \in [D\mathcal{V}(\vartheta), D\mathcal{V}(\vartheta + \delta_F)] \ \frac{\delta_F}{\mathcal{V}_R},$$

and

$$\mathcal{M}(\vartheta) \geqslant \frac{\delta_F}{\mathcal{V}_R} D\mathcal{V}(\vartheta) + \tau + \beta.$$

Now define

$$G(\vartheta) := \sin(\vartheta + \gamma\delta_F) \left(\frac{\delta_F}{\mathcal{V}_R} D\mathcal{V}(\vartheta) + \tau + \beta\right)^{\frac{1}{1-d}}, \tag{3.11.24}$$

Then, since $\delta_F < \gamma\delta_F$, we have, for $\vartheta \in [\vartheta_{F,2}, \frac{\pi}{2} - \gamma\delta_F]$,

$$Q(\vartheta) \leqslant G(\vartheta). \tag{3.11.25}$$

We use the abbreviation

$$\chi := \frac{\delta_F}{\mathcal{V}_R} \omega_{d-1}, \tag{3.11.26}$$

and define

$$S(\vartheta) := \sin^{d-1}(\vartheta + \gamma\delta_F) \left(\chi \sin^{d-1} \vartheta - 1\right)^{-1}. \tag{3.11.27}$$

By (3.5.71) we have

$$\chi \ \sin^{d-1} \vartheta - 1 \geqslant \chi \ \sin^{d-1} \vartheta_{F,2} - 1 > \frac{5}{2} > 0.$$

From (3.5.39) we know that $|\tau+\beta| \leqslant 1$. Therefore by (2.3.41) and (2.2.6) we have

$$\begin{aligned} G(\vartheta)^{d-1} &= \sin^{d-1}(\vartheta+\gamma\delta_F)\left(\frac{\delta_F}{\mathcal{V}_R}D\mathcal{V}(\vartheta)+\tau+\beta\right)^{-1} \\ &= \sin^{d-1}(\vartheta+\gamma\delta_F)\left(\frac{\delta_F}{\mathcal{V}_R}\omega_{d-1}\ \sin^{d-1}\vartheta+\tau+\beta\right)^{-1} \\ &\leqslant \sin^{d-1}(\vartheta+\gamma\delta_F)\left(\chi\ \sin^{d-1}\vartheta-1\right)^{-1} = S(\vartheta). \end{aligned} \tag{3.11.28}$$

To determine if $S(\vartheta)$ is monotonic in ϑ, we differentiate and find that

$$\begin{aligned} \frac{\partial}{\partial\vartheta}S(\vartheta) &= \frac{\partial}{\partial\vartheta}\big(\sin^{d-1}(\vartheta+\gamma\delta_F)(\chi\ \sin^{d-1}\vartheta-1)^{-1}\big) \\ &= \frac{\partial}{\partial\vartheta}\big(\sin^{d-1}(\vartheta+\gamma\delta_F)\big)(\chi\ \sin^{d-1}\vartheta-1)^{-1} \\ &\quad+\sin^{d-1}(\vartheta+\gamma\delta_F)\frac{\partial}{\partial\vartheta}\big((\chi\ \sin^{d-1}\vartheta-1)^{-1}\big) \\ &= (d-1)\ \sin^{d-2}(\vartheta+\gamma\delta_F)\cos(\vartheta+\gamma\delta_F)(\chi\ \sin^{d-1}\vartheta-1)^{-1} \\ &\quad+\sin^{d-1}(\vartheta+\gamma\delta_F)\big(-(\chi\ \sin^{d-1}\vartheta-1)^{-2}(d-1)\chi\ \sin^{d-2}\vartheta\ \cos\vartheta\big) \\ &= (d-1)\ \sin^{d-2}(\vartheta+\gamma\delta_F)(\chi\ \sin^{d-1}\vartheta-1)^{-2} \\ &\quad\left(\cos(\vartheta+\gamma\delta_F)(\chi\ \sin^{d-1}\vartheta-1)-\sin(\vartheta+\gamma\delta_F)\ \chi\ \sin^{d-2}\vartheta\ \cos\vartheta\right). \end{aligned}$$

But

$$\begin{aligned} &\cos(\vartheta+\gamma\delta_F)(\chi\ \sin^{d-1}\vartheta-1)-\sin(\vartheta+\gamma\delta_F)\ \chi\ \sin^{d-2}\vartheta\ \cos\vartheta \\ &= \cos(\vartheta+\gamma\delta_F)\ \chi\ \sin^{d-2}\vartheta\ \sin\vartheta-\sin(\vartheta+\gamma\delta_F)\ \chi\ \sin^{d-2}\vartheta\ \cos\vartheta-\cos(\vartheta+\gamma\delta_F) \\ &= \chi\ \sin^{d-2}\vartheta\ \big(\cos(\vartheta+\gamma\delta_F)\sin\vartheta-\sin(\vartheta+\gamma\delta_F)\cos\vartheta\big)-\cos(\vartheta+\gamma\delta_F) \\ &= -\chi\ \sin^{d-2}\vartheta\ \sin\gamma\delta_F-\cos(\vartheta+\gamma\delta_F)<0. \end{aligned}$$

So $S(\vartheta)$ is monotonically decreasing with ϑ in the domain $\mathbb{D}_{mL}$. Therefore

$$S(\vartheta)\leqslant S(\vartheta_{F,2}). \tag{3.11.29}$$

Finally, using (3.11.22), (3.11.25), (3.11.28) and (3.11.29) we have

$$\mathcal{P}(\vartheta) \leqslant S(\vartheta_{F,2})^{\frac{1}{d-1}}.$$

Estimate for $\mathbb{D}_{mL}$.

From (3.11.27) we obtain

$$\begin{aligned}
\mathcal{P}(\vartheta) \leqslant S(\vartheta_{F,2})^{\frac{1}{d-1}} &= \sin(\vartheta_{F,(3+\eta)}) \left(\chi \sin^{d-1} \vartheta_{F,2} - 1\right)^{\frac{1}{1-d}} \\
&\leqslant \vartheta_{F,(3+\eta)} \left(\chi \sin^{d-1} \vartheta_{F,2} - 1\right)^{\frac{1}{1-d}} \\
&= \vartheta_{F,(3+\eta)} \chi^{\frac{1}{1-d}} (\sin \vartheta_{F,2})^{-1} \left(1 - \frac{1}{\chi \sin^{d-1} \vartheta_{F,2}}\right)^{\frac{1}{1-d}}.
\end{aligned}$$

Motivated by (3.5.57) define

$$L(x) := J_{F,2}(x) \left(\left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}} + \rho_L(x) \right) \tag{3.11.30}$$

so that $\sin \vartheta_{F,2} \geqslant L(x) \delta_I$. This estimate requires that

$$\Theta\left(\frac{\omega_d}{x}\right) + \rho_H(x) \left(\frac{\omega_d}{x}\right)^{\frac{1}{d}} \leqslant \frac{\pi}{2}.$$

This is implied by the stronger condition (3.11.14), which we assume from here onwards. Since $J_{F,2}(x) \nearrow 1$ and $\rho_L(x) \nearrow 1$ as $x \to \infty$, we have

$$L(x) \nearrow L(\infty) := \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + 1 \tag{3.11.31}$$

as $x \to \infty$.

Recall from (3.11.26) that

$$\chi = \frac{\delta_F}{\mathcal{V}_R} \; \omega_{d-1} = \rho \delta_I^{1-d} \; \omega_{d-1}.$$

We then have

$$\chi \sin^{d-1} \vartheta_{F,2} \geqslant \rho_L(x)\ \delta_I^{1-d}\ \omega_{d-1} L(x)^{d-1} \delta_I^{d-1} = \rho_L(x)\ \omega_{d-1} L(x)^{d-1},$$

so that, using (3.5.54), we have

$$\begin{aligned}\mathcal{P}(\vartheta) &\leqslant \vartheta_{F,(3+\eta)}\ \chi^{\frac{1}{1-d}} (\sin \vartheta_{F,2})^{-1} \ \left(1 - \frac{1}{\chi \sin^{d-1} \vartheta_{F,2}}\right)^{\frac{1}{1-d}} \\ &\leqslant \frac{\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (2+\eta)\rho_H(x)}{\left(\rho_L(x)\ \omega_{d-1}\right)^{\frac{1}{d-1}}\ L(x)} \ \left(1 - \frac{L(x)^{1-d}}{\rho_L(x)\ \omega_{d-1}}\right)^{\frac{1}{1-d}} \ \delta_I.\end{aligned}$$

The estimate above requires that

$$\rho_L(x)\ \omega_{d-1} L(x)^{d-1} > 1,$$

but this condition is (3.5.63), which we have already assumed. We know that $J_c(x) \nearrow 1$, $\rho_H(x) \searrow 1$, $\rho_L(x) \nearrow 1$ and from (3.11.31) we have $L(x) \nearrow L(\infty)$. We therefore see that

$$\begin{aligned}&\frac{\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (2+\eta)\rho_H(x)}{\left(\rho_L(x)\ \omega_{d-1}\right)^{\frac{1}{d-1}}\ L(x)} \ \left(1 - \frac{L(x)^{1-d}}{\rho_L(x)\ \omega_{d-1}}\right)^{\frac{1}{1-d}} \\ &\qquad \searrow \frac{\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + (2+\eta)}{\omega_{d-1}^{\frac{1}{d-1}}\ L(\infty)} \ \left(1 - \frac{L(\infty)^{1-d}}{\omega_{d-1}}\right)^{\frac{1}{1-d}}.\end{aligned}$$

Assuming that $\mathcal{N} > x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and that x satisfies (3.5.62), (3.5.63) and (3.11.14), the resulting estimate is then

$$\mathcal{P}(\vartheta) \leqslant \mathsf{C}_{mL}(x) \mathcal{N}^{-\frac{1}{d}},$$

where

$$\mathsf{C}_{mL}(x) := \frac{\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + (2+\eta)\rho_H(x)}{\left(\rho_L(x)\ \omega_{d-1}\right)^{\frac{1}{d-1}}\ L(x)} \ \left(1 - \frac{L(x)^{1-d}}{\rho_L(x)\ \omega_{d-1}}\right)^{\frac{1}{1-d}} \ \omega_d^{\frac{1}{d}}, \tag{3.11.32}$$

with

$$\mathsf{C}_{mL}(x) \searrow \mathsf{C}_{mL}(\infty) := \frac{\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} + (2+\eta)}{\omega_{d-1}^{\frac{1}{d-1}}\ L(\infty)} \ \left(1 - \frac{L(\infty)^{1-d}}{\omega_{d-1}}\right)^{\frac{1}{1-d}} \ \omega_d^{\frac{1}{d}} \tag{3.11.33}$$

as $x \to \infty$.

The domain $\mathbb{D}_{mH}$.

Here we have $\vartheta \in \left(\frac{\pi}{2} - \gamma\delta_F, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$. We know from (3.5.25) that

$$\begin{aligned}
\mathcal{P}(\vartheta) &= \mathcal{W}(\vartheta)\,\mathcal{M}(\vartheta)^{\frac{1}{1-d}} \\
&= \left(\max_{\xi \in [\mathcal{T}(\tau,\vartheta), \mathcal{B}(\beta,\vartheta)]} \sin\xi\right) \mathcal{M}(\vartheta)^{\frac{1}{1-d}} \\
&\leqslant \mathcal{M}(\vartheta)^{\frac{1}{1-d}}.
\end{aligned} \tag{3.11.34}$$

The case $\mathbb{D}_{mH}$ now splits temporarily into two sub-cases.

1. For $\vartheta \in \left(\frac{\pi}{2} - \gamma\delta_F, \frac{\pi}{2} - \delta_F\right]$, (2.3.42) yields

$$\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta) \in \left[\delta_F D\mathcal{V}(\vartheta), \delta_F D\mathcal{V}(\vartheta + \delta_F)\right].$$

2. For $\vartheta \in \left(\frac{\pi}{2} - \delta_F, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$, we know from (2.3.44) that

$$\mathcal{V}(\vartheta + \delta_F) = \omega_d - \mathcal{V}(\pi - \vartheta - \delta_F) \quad \text{and} \quad \mathcal{V}\left(\frac{\pi}{2}\right) = \frac{\omega_d}{2}.$$

   Therefore

$$\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}\left(\frac{\pi}{2}\right) = \mathcal{V}\left(\frac{\pi}{2}\right) - \mathcal{V}(\pi - \vartheta - \delta_F).$$

   This means that the portion of the interval $[\vartheta, \vartheta + \delta_F]$ which lies in the Southern hemisphere is equivalent to the interval $\left[\pi - \vartheta - \delta_F, \frac{\pi}{2}\right]$ in the Northern hemisphere. If we now define $\overline{\vartheta} := \pi - \vartheta - \delta_F$, we have $\overline{\vartheta} \in \left[\frac{\pi}{2} - \delta_F, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$, and therefore

$$\begin{aligned}\mathcal{V}(\vartheta+\delta_F)-\mathcal{V}(\vartheta) &= \mathcal{V}\left(\frac{\pi}{2}\right)-\mathcal{V}(\overline{\vartheta})+\mathcal{V}\left(\frac{\pi}{2}\right)-\mathcal{V}(\vartheta)\\ &\geqslant \left(\frac{\pi}{2}-\overline{\vartheta}\right)D\mathcal{V}(\overline{\vartheta})+\left(\frac{\pi}{2}-\vartheta\right)D\mathcal{V}(\vartheta)\\ &\geqslant \left(\pi-(\overline{\vartheta}+\vartheta)\right)\min\left(D\mathcal{V}(\overline{\vartheta}),D\mathcal{V}(\vartheta)\right)\geqslant \delta_F D\mathcal{V}\left(\frac{\pi}{2}-\delta_F\right).\end{aligned}$$

The last inequality is justified by Lemma 2.3.16.

We can now put the two sub-cases back together by noting that

$$D\mathcal{V}\left(\frac{\pi}{2}-\gamma\delta_F\right)<D\mathcal{V}\left(\frac{\pi}{2}-\delta_F\right),$$

since $\gamma\delta_F>\delta_F$. In other words

$$\mathcal{V}(\vartheta+\delta_F)-\mathcal{V}(\vartheta)\geqslant\delta_F D\mathcal{V}(\frac{\pi}{2}-\gamma\delta_F).$$

Using (2.3.41), (3.5.23), (3.5.39) and (3.11.26), we therefore have

$$\begin{aligned}\mathcal{M}(\vartheta) &\geqslant \chi\sin^{d-1}\left(\frac{\pi}{2}-\gamma\delta_F\right)+\tau+\beta\\ &\geqslant \chi\sin^{d-1}\left(\frac{\pi}{2}-\gamma\delta_F\right)-1\\ &\geqslant \chi\left(1-\frac{2}{\pi}\gamma\delta_F\right)^{d-1}-1\\ &\geqslant \frac{\rho_L(x)\delta_I}{\mathcal{V}_R}\omega_{d-1}\left(1-\frac{2}{\pi}\gamma\ \rho_H(x)\delta_I\right)^{d-1}-1\\ &\geqslant \frac{\rho_L(x)\delta_I}{\mathcal{V}_R}\omega_{d-1}\left(1-\frac{2}{\pi}\gamma\ \rho_H(x)\ \omega_d^{\frac{1}{d}}\ x^{-\frac{1}{d}}\right)^{d-1}-1.\end{aligned}$$

Therefore

$$\mathcal{M}(\vartheta)\geqslant\Lambda(x)\mathcal{N}^{\frac{d-1}{d}}-1, \tag{3.11.35}$$

where

$$\Lambda(x) := \rho_L(x)\ \omega_d^{\frac{1-d}{d}}\ \omega_{d-1} \left(1 - \frac{2}{\pi}\ \gamma\ \rho_H(x)\ \omega_d^{\frac{1}{d}}\ x^{-\frac{1}{d}}\right)^{d-1}. \tag{3.11.36}$$

Here we need to assume that

$$\frac{2}{\pi}\ \gamma\ \rho_H(x)\ \omega_d^{\frac{1}{d}}\ x^{-\frac{1}{d}} < 1,$$

so that $\Lambda(x) > 0$. This occurs when

$$\frac{x}{\rho_H(x)^d} > \left(\frac{2\gamma}{\pi}\right)^d \omega_d. \tag{3.11.37}$$

This holds for x sufficiently large, since $\rho_H(x) \searrow 1$ as $x \to \infty$.

In fact we see that since $\rho_L(x) \nearrow 1$ and $\rho_H(x) \searrow 1$ as $x \to \infty$,

$$\Lambda(x) \nearrow \Lambda(\infty) := \omega_d^{\frac{1-d}{d}}\ \omega_{d-1} \tag{3.11.38}$$

as $x \to \infty$.

Estimate for $\mathbb{D}_{mH}$.

Assume that $\mathcal{N} > x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and that x satisfies (3.5.62), (3.5.63) and (3.11.37).

First, we see that

$$\Lambda(x)\mathcal{N}^{\frac{d-1}{d}} - 1 \geqslant \left(\Lambda(x) - x^{\frac{1-d}{d}}\right)\mathcal{N}^{\frac{d-1}{d}},$$

We want this last expression to be positive. This occurs when $x > \Lambda(x)^{\frac{d}{d-1}}$, that is, when

$$x > \omega_d^{-1}\ \left(\rho_L(x)\ \omega_{d-1}\right)^{\frac{d}{d-1}} \left(1 - \frac{2}{\pi}\ \gamma\ \rho_H(x)\ \omega_d^{\frac{1}{d}}\ x^{-\frac{1}{d}}\right)^d. \tag{3.11.39}$$

Since

$$\Lambda(x)^{\frac{d}{d-1}} \nearrow \Lambda(\infty)^{\frac{d}{d-1}} = \omega_d^{-1}\ \omega_{d-1}^{\frac{d}{d-1}}$$

as $x \to \infty$, we see that (3.11.39) holds for x sufficiently large.

We now have from (3.11.34) and (3.11.35) that for $\mathcal{N} > x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x satisfies (3.5.62), (3.5.63), (3.11.37) and (3.11.39) the estimate

$$\mathcal{P}(\vartheta) \leqslant \mathcal{M}(\vartheta)^{\frac{1}{1-d}} \leqslant \mathsf{C}_{mH}(x)\mathcal{N}^{-\frac{1}{d}},$$

where

$$\mathsf{C}_{mH}(x) := \left(\Lambda(x) - x^{\frac{1-d}{d}}\right)^{\frac{1}{1-d}}, \tag{3.11.40}$$

and where, considering (3.11.38), we have

$$\mathsf{C}_{mH}(x) \searrow \mathsf{C}_{mH}(\infty) := \Lambda(\infty)^{\frac{1}{1-d}} = \omega_d^{\frac{1}{d}}\ \omega_{d-1}^{\frac{1}{1-d}} \tag{3.11.41}$$

as $x \to \infty$.

Final result.

Given $x \geqslant \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), such that x satisfies (3.5.62), (3.5.63), (3.11.14), (3.11.37) and (3.11.39), the following estimate obtains.

For $\mathcal{N} > x$ and $(\tau, \beta, \vartheta) \in \mathbb{D}_+$, we have

$$\mathcal{P}(\vartheta) \leqslant \mathsf{C}_P(x)\mathcal{N}^{-\frac{1}{d}}, \quad \text{where} \quad \mathsf{C}_P(x) := \max(\mathsf{C}_t(x), \mathsf{C}_{mL}(x), \mathsf{C}_{mH}(x)), \tag{3.11.42}$$

with $\mathsf{C}_P(x)$ monotonic decreasing in x. This completes the proof of Lemma 3.5.26.

$\square$

3.11.3 A per-region bound on diameter

Proof of Theorem 3.7.2.

For the whole sphere $\mathbb{S}^d$, we have

$$\operatorname{diam} \mathbb{S}^d = 2 = \operatorname{db} \mathbb{S}^d.$$

The partition algorithm for $\mathrm{EQ}(1, \mathcal{N})$, with $\mathcal{N} > 1$, divides $\mathbb{S}^1$ into $\mathcal{N}$ equal segments, as described in Section 3.2.1. For a region R contained in $\mathrm{EQ}(1, \mathcal{N})$, with $\mathcal{N} > 1$, the region can be therefore be described by the pair of polar coordinates α, β. That is, $\mathsf{R} = \mathcal{R}(\alpha, \beta)$. The spherical distance $s(\odot(\alpha), \odot(\beta))$ is then given by

$$\mathsf{s}(\odot(\alpha), \odot(\beta)) = \frac{2\pi}{\mathcal{N}} \leqslant \pi.$$

Using Lemma 2.3.2, the diameter of R is then

$$\operatorname{diam} \mathsf{R} = \|\odot(\alpha) - \odot(\beta)\| = \Upsilon\left(\mathsf{s}(\odot(\alpha), \odot(\beta))\right) = \Upsilon\left(\frac{2\pi}{\mathcal{N}}\right) = \operatorname{db} \mathsf{R}.$$

For $d > 1$, for a spherical cap R with spherical radius ϑ_c, by Lemma 3.3.1,

$$\operatorname{diam} \mathsf{R} = 2 \sin \vartheta_c = \operatorname{db} \mathsf{R}.$$

For $d > 1$, for a region R contained in collar $i \in \{1, \ldots, \mathsf{n}\}$ of a recursive zonal equal area partition of $\mathbb{S}^d$ with n collars, by Lemma 3.3.5, if $\operatorname{diam} \Pi\, \mathsf{R} \leqslant \operatorname{db} \Pi\, \mathsf{R}$ then

$$\begin{aligned}\operatorname{diam} \mathsf{R} &\leqslant \sqrt{\Upsilon(\delta_i)^2 + \mathsf{w}_i^2 (\operatorname{diam} \Pi\, \mathsf{R})^2} \\ &\leqslant \sqrt{\Upsilon(\delta_i)^2 + \mathsf{w}_i^2 (\operatorname{db} \Pi\, \mathsf{R})^2} = \operatorname{db} \mathsf{R}.\end{aligned}$$

The result follows by induction on d. □

3.11.4 The Feige–Schechtman lemma

Proof of Lemma 3.8.1.

This proof essentially follows [54, Lemma 21, p. 430-431].

1. Given $d > 1$, $\mathcal{N} > 2$, use (3.2.2) to determine ϑ_c. Then we have $\mathcal{V}(\vartheta_c) = \mathcal{V}_R = \omega_d/\mathcal{N}$, with $\mathcal{V}_R$ being the area we need for each region of the partition.

2. Now create a saturated packing of $\mathbb{S}^d$ by caps of spherical radius ϑ_c, as per Definition 2.7.3, constructed via a greedy algorithm so that each cap kisses at least one other cap. Let m be the number of caps in the packing.

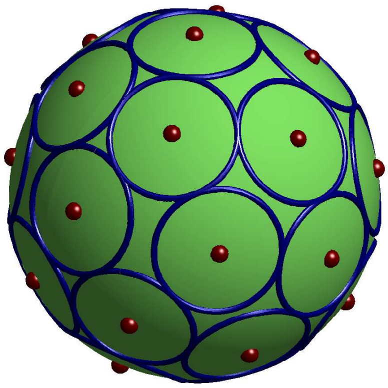

Figure 3.8: Step 2 of the Feige-Schechtman construction

(Figure 3.8 uses the putatively optimal packing of 27 points on $\mathbb{S}^2$ as found by Tarnai and Gáspár [151, pp. 205–206], and listed by Kottwitz [87, Table 1, p. 161] and Sloane [143, pack.3.27.txt]. This packing is used for illustration purposes only.)

We see that no point of $\mathbb{S}^d$ is more than $2\vartheta_c$ from the centre of a cap, otherwise we could have added another cap. Therefore the m centre points of the packing are also the centres of a covering of $\mathbb{S}^d$ by spherical caps of spherical radius $2\vartheta_c$ [164, p. 1091] [165, Lemma 1, p. 2112]. (See Figure 3.9, where the boundaries of the covering caps are shown in yellow.)

3. Now partition $\mathbb{S}^d$ into Voronoi cells V_i, $i \in \{1, \ldots, m\}$ based on these m centre points. The Voronoi cell V_i corresponding to centre point i consists of those

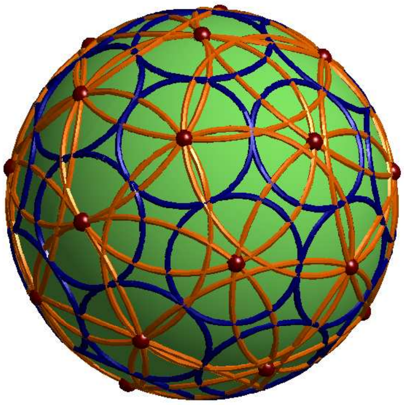

Figure 3.9: Step 2 of the Feige-Schechtman construction, showing covering caps

points of $\mathbb{S}^d$ which are at least as close to the centre point i as they are to of any of the other centre points. (See Figure 3.10.)

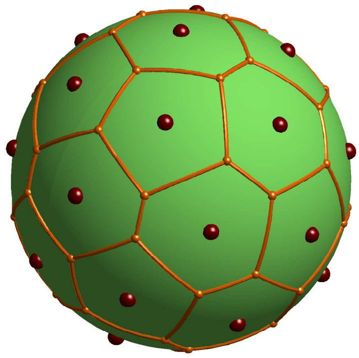

Figure 3.10: Step 3 of the Feige-Schechtman construction

We see that the Voronoi cells must contain the packing caps and be contained in the covering caps. Thus each V_i has area at least $\mathcal{V}_R$ and spherical diameter at most $\min(\pi, 2\vartheta_c)$.

4. Now create a graph Γ with a node for each centre point and an edge for each pair of kissing packing caps. (See Figure 3.11.)
5. Take any spanning tree S of Γ (also known as a *maximal tree* [116, Section 6.2 pp. 101–103]). (See Figure 3.12.)

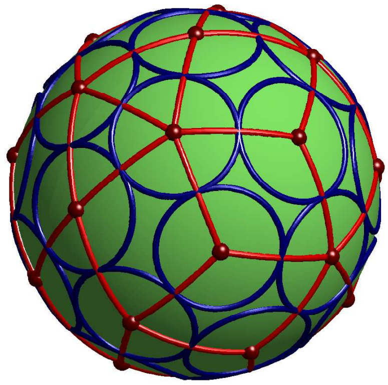

Figure 3.11: Step 4 of the Feige-Schechtman construction

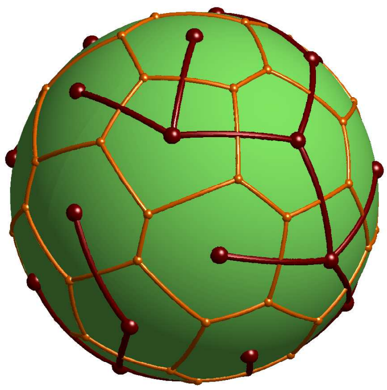

Figure 3.12: Step 5 of the Feige-Schechtman construction

The tree S has leaves, which are nodes having only one edge, and either a single centre node, or a bicentre, which is a pair of nodes joined by an edge. The centre or bicentre nodes are the nodes for which the shortest path to any leaf has the maximum number of edges [27] [28, Volume 9, p. 430] [128, Chapter 6, Section 9, p. 135]. If there is a single centre, mark it as the root node. If there is a bicentre, arbitrarily mark one of the two nodes as the root node. Now create the directed tree T from S by directing the edges from the leaves towards the root [128, Chapter 6, Section 7, p. 129].

6. For each leaf j, of T define $n_j := \lfloor \sigma(V_j)/\mathcal{V}_R \rfloor$, (with $\lfloor x \rfloor$ denoting the least integer function of x).
7. Partition V_j into the super-region U_j with $\sigma(U_j) = n_j \mathcal{V}_R$ and the remainder $W_j := V_j \setminus U_j$.

8. For each nonleaf node k other than the root, define $X_k = V_k \cup \bigcup_{(j,k)\in T} W_j$, that is, we add all the remainders of the daughters of k to V_k to obtain X_k.
9. Now define $n_k := \lfloor \sigma(X_k)/\mathcal{V}_R \rfloor$ and partition X_k into the super-region U_k with $\sigma(U_k) = n_k \mathcal{V}_R$ and the remainder $W_k := X_k \setminus U_k$.
10. Continue until only the root node is left.
11. For the root node ℓ, if we define $U_\ell := V_\ell \cup \bigcup_{(k,\ell)\in T} W_k$, we see that we must have $\sigma(U_\ell) = n_\ell \mathcal{V}_R$, where

$$n_\ell := \mathcal{N} - \sum_{i \neq \ell} n_i.$$

that is, the area of the super-region corresponding to the root node must be an integer multiple of $\mathcal{V}_R$.

Since at each step we have assembled U_i only from the Voronoi cells corresponding to kissing packing caps, each U_i is contained in a spherical cap with centre the same as the centre of the corresponding packing cap, and spherical radius $\min(\pi, 4\vartheta_c)$, and so the spherical diameter of each U_i is at most $\min(\pi, 8\vartheta_c)$.

12. Now partition each U_i into n_i regions of area $\mathcal{V}_R$, and let $FS(d,\mathcal{N})$ be the resulting partition of $\mathbb{S}^d$. Then $FS(d,\mathcal{N})$ is a partition of $\mathbb{S}^d$ into $\mathcal{N}$ regions, with each region $\mathsf{R} \in FS(d,\mathcal{N})$ having area $\omega_d/\mathcal{N}$ and Euclidean diameter bounded above by

$$\operatorname{diam} \mathsf{R} \leqslant \Upsilon\left(\min(\pi, 8\vartheta_c)\right) = 2\sin\left(\min\left(\frac{\pi}{2}, 4\vartheta_c\right)\right).$$

$\square$

Remarks. Feige and Schechtman's proof [54, Lemma 21, p. 430-431] uses a maximal packing instead of a saturated packing, but maximality is harder to achieve and the proof of Lemma 3.8.1 only needs a saturated packing.

3.12 Approximate values of constants

We here tabulate the approximate values of N_d and K_d as calculated by Maple using the various definitions given in the proofs of lemmas in Section 3.11 and the proof of Theorem 3.1.3. The values depend on the choice of $N_H \geqslant x > \mathcal{N}_0(5)$, where $\mathcal{N}_0$ is defined by (3.5.8), and x is further constrained by (3.5.62), (3.5.63), (3.11.14), (3.11.37) and (3.11.39).

First, we try setting N_H to be $\lceil x \rceil$, where x satisfies all the necessary constraints. We then use Maple to calculate the constants listed in Table 3.1.

d	N_H	K_L	K_Δ	C_t	C_{mL}	C_{mH}	C_P	K_H	K_d
2	7.90×10^1	17.78	4.91	1.15	1.20	0.98	1.20	9.01	17.78
3	1.56×10^2	10.77	3.69	1.54	1.52	1.37	1.54	27.54	27.54
4	4.41×10^2	9.17	3.02	1.60	1.61	1.42	1.61	44.45	44.45
5	1.13×10^3	8.16	2.60	1.63	1.66	1.44	1.66	73.74	73.74
6	2.70×10^3	7.46	2.31	1.63	1.69	1.45	1.69	124.60	124.60
7	6.78×10^3	7.05	2.10	1.62	1.70	1.44	1.70	211.99	211.99
8	2.26×10^4	7.00	1.92	1.57	1.68	1.40	1.68	356.91	356.91
9	7.89×10^4	7.00	1.77	1.52	1.67	1.36	1.67	595.28	595.28
10	2.87×10^5	7.03	1.66	1.48	1.65	1.33	1.65	985.08	985.08

Table 3.1: Constants from proof of Theorem 3.1.3

We now improve the constants above in two ways. First, for $d \leqslant 4$ and $\mathcal{N} \leqslant 100\ 000$, we estimate $\operatorname{maxdiam}(d, \mathcal{N})$ by computing the per region diameter bound for each region of $\mathrm{EQ}(d, \mathcal{N})$ as per Sections 5 and 8 of [99]. We then set $N_H = 100\ 000$ for these values of d and set K_L to be the maximum estimated value of $\operatorname{maxdiam}(d, \mathcal{N})$ obtained.

Second, for $d > 4$, we use Maple to find $N_H > x$ such that x satisfies all the necessary constraints and such that $K_L = K_H$. Maple then obtains the constants listed in Table 3.2. In particular, for this choice of N_H, we have $K_2 < 7.4$ and $K_3 < 9.1$. Zhou obtains $K_2 \leqslant 7$ for his algorithm [167, Theorem 2.8 p 13]. The larger bound here can be explained by the crudeness of the approximations used to prove the lemmas and Theorem 3.1.3, and in particular, the use of separate bounds for K_Δ

d	N_H	K_L	K_Δ	C_t	C_{mL}	C_{mH}	C_P	K_H	K_d	K_{FS}
2	1.00×10^5	< 6.5	4.55	0.87	0.92	0.57	0.92	7.37	7.37	16.00
3	1.00×10^5	< 7.0	3.36	1.09	1.14	0.80	1.14	9.06	9.06	13.41
4	1.00×10^5	< 7.5	2.78	1.18	1.26	0.93	1.26	11.77	11.77	12.16
5	3.82×10^4	16.50	2.45	1.29	1.39	1.07	1.39	16.50	16.50	11.41
6	2.04×10^6	22.52	2.14	1.22	1.36	1.02	1.36	22.52	22.52	10.90
7	1.92×10^8	30.50	1.93	1.19	1.35	1.00	1.35	30.50	30.50	10.54
8	3.29×10^{10}	41.28	1.76	1.17	1.35	0.98	1.35	41.28	41.28	10.26
9	1.07×10^{13}	56.05	1.64	1.16	1.36	0.97	1.36	56.05	56.05	10.04
10	6.71×10^{15}	76.51	1.53	1.14	1.36	0.97	1.36	76.51	76.51	9.86

Table 3.2: Improved constants from proof of Theorem 3.1.3

and C_P over $\mathbb{D}$. We have added the Feige–Schechtman constant K_{FS} for comparison and we see that $K_d < K_{FS}$ for $d \leqslant 4$.

If we ignore K_L and take the limit of K_H as $N_H \to \infty$, Maple obtains the constants listed in Table 3.3.

d	$K_\Delta(\infty)$	$\mathsf{C}_t(\infty)$	$\mathsf{C}_{mL}(\infty)$	$\mathsf{C}_{mH}(\infty)$	$\mathsf{C}_P(\infty)$	$K_d(\infty)$	K_{FS}
2	4.54	0.87	0.92	0.56	0.92	7.34	16.00
3	3.32	1.07	1.12	0.76	1.12	8.86	13.41
4	2.72	1.13	1.21	0.84	1.21	11.02	12.16
5	2.34	1.15	1.25	0.88	1.25	14.00	11.41
6	2.08	1.15	1.28	0.90	1.28	18.10	10.90
7	1.89	1.15	1.31	0.92	1.31	23.71	10.54
8	1.74	1.15	1.32	0.93	1.32	31.43	10.26
9	1.62	1.14	1.34	0.94	1.34	42.11	10.04
10	1.52	1.14	1.35	0.94	1.35	56.99	9.86

Table 3.3: Limiting constants from proof of Theorem 3.1.3

Chapter 4

Spherical codes based on equal area partitions

> "That's a thread that has linked several recent projects: we (usually Hardin and me) run the computer to find good packings, or coverings, or designs, we stare hard at the results, we learn, and we generalize."
> – Neil Sloane interviewed by A. R. Calderbank in [48].

We use the notation $\mathrm{EQP}(d,\mathcal{N})$ to denote the recursive zonal equal area (EQ) code with $\mathcal{N}$ codepoints in the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{+1}$.

4.1 Construction of the EQ codes

In essence, $\mathrm{EQP}(d,\mathcal{N})$ is constructed by taking the partition $\mathrm{EQ}(d,\mathcal{N})$ and placing one point, called a *codepoint* at the "centre" of each region of the partition. There are two areas of ambiguity in this construction, which will be discussed and removed in this section.

Figure 4.1 illustrates the code $\mathrm{EQP}(2,33)$ in red, with the boundaries of the partition $\mathrm{EQ}(2,33)$ shown in blue.

4.1.1 Exactly where is the centre *of a region?*

First, we need to specify what is meant by the "centre" of a region. Recalling (3.2.12), each region R of $\mathrm{EQ}(d,\mathcal{N})$ is of the form

$$\mathsf{R} = \mathcal{R}\left((\tau_1,\ldots,\tau_d),(\upsilon_1,\ldots,\upsilon_d)\right) = \odot\left([\tau_1,\upsilon_1] \times \ldots \times [\tau_d,\upsilon_d]\right).$$

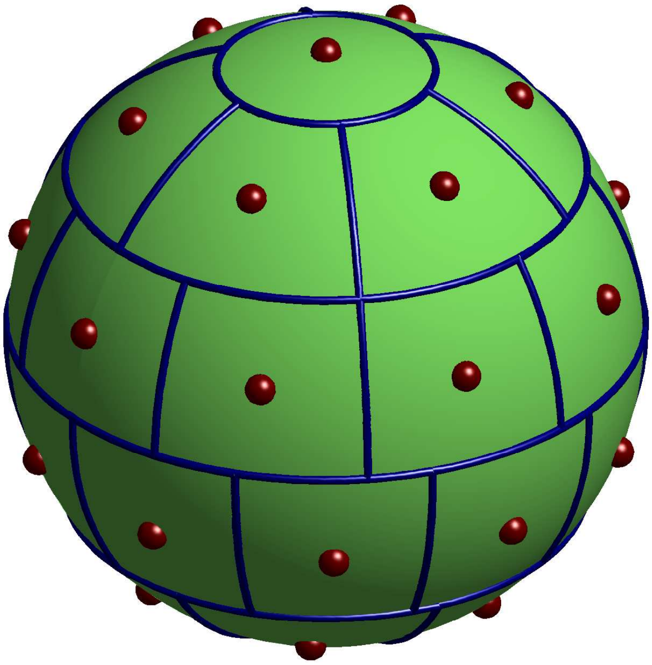

Figure 4.1: EQ code $\mathrm{EQP}(2,33)$, showing the partition $\mathrm{EQ}(2,33)$

In particular, the North polar cap is of the form

$$\mathsf{R}_1 = \odot\big([0,2\pi] \times [0,\pi] \times \ldots \times [0,\pi] \times [0,\vartheta_c]\big),$$

and the South polar cap is of the form

$$\mathsf{R}_{\mathcal{N}} = \odot\big([0,2\pi] \times [0,\pi] \times \ldots \times [0,\pi] \times [\pi-\vartheta_c,\pi]\big).$$

In general, because of the recursive nature of the construction, some regions of $\mathrm{EQ}(d,\mathcal{N})$ may be descendants of a circle, having the form

$$\mathsf{R} = \odot\big([0,2\pi] \times [\tau_2,\upsilon_2] \times \ldots \times [\tau_d,\upsilon_d]\big),$$

or may be descendants of a polar cap, having the form

$$\mathsf{R} = \odot\big([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [0, \upsilon_k] \times [\tau_{k+1}, \upsilon_{k+1}] \times \ldots \times [\tau_d, \upsilon_d]\big),$$

or

$$\mathsf{R} = \odot\big([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [\tau_k, \pi] \times [\tau_{k+1}, \upsilon_{k+1}] \times \ldots \times [\tau_d, \upsilon_d]\big).$$

In fact, the interval of longitude may be of the form $[\tau_1, \tau_1 + 2\pi] \pmod{2\pi}$.

The algorithm to determine the spherical coordinates of the codepoint

$$\mathbf{a} = \odot(\alpha_1, \ldots, \alpha_d) = \boxdot\,\mathcal{R}(\tau, \upsilon) := \boxdot(\tau, \upsilon)$$

in terms of the pseudo-vertices τ and υ, can be written in pseudocode as:

if $\upsilon_1 = \tau_1 + 2\pi \pmod{2\pi}$ **then**

$\alpha_1 := 0$;

else $\alpha_1 := (\tau_1 + \upsilon_1)/2 \pmod{2\pi}$;

endif;

for $k \in \{2, \ldots, d\}$ **do**

if $\tau_k = 0$ **then** $\alpha_k := 0$;

else if $\upsilon_k = \pi$ **then** $\alpha_k := \pi$;

else $\alpha_k := (\tau_k + \upsilon_k)/2$;

endif;

enddo.

Using this algorithm, we see that

$$\mathsf{R}_1 = \odot\big([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [0, \upsilon_k] \times [\tau_{k+1}, \upsilon_{k+1}] \times \ldots \times [\tau_d, \upsilon_d]\big)$$

yields

$$\boxdot\, \mathsf{R}_1 = \odot\left(0, 0, \ldots, 0, 0, \frac{\tau_{k+1} + \upsilon_{k+1}}{2}, \ldots, \frac{\tau_d + \upsilon_d}{2}\right),$$

while

$$\mathsf{R}_2 = \odot\left([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [\tau_k, \pi] \times [\tau_{k+1}, \upsilon_{k+1}] \times \ldots \times [\tau_d, \upsilon_d]\right)$$

yields

$$\boxdot\, \mathsf{R}_2 = \odot\left(0, 0, \ldots, 0, \pi, \frac{\tau_{k+1} + \upsilon_{k+1}}{2}, \ldots, \frac{\tau_d + \upsilon_d}{2}\right),$$

and in particular, a North polar cap yields the North Pole,

$$\boxdot \odot\left([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [0, \upsilon_d]\right) = \odot(0, 0, \ldots, 0, 0),$$

while a South polar cap yields the South pole,

$$\boxdot \odot\left([0, 2\pi] \times [0, \pi] \times \ldots \times [0, \pi] \times [\tau_d, \pi]\right) = \odot(0, 0, \ldots, 0, \pi).$$

4.1.2 Exactly where are the regions?

As noted in Section 3.2, the partition $\mathrm{EQ}(d, \mathcal{N})$ is not fully specified by the algorithm described there. The algorithm instead specifies an equivalence class of partitions, unique up to rotations of the partitions of $\mathbb{S}^1$. This means that the collars of $\mathrm{EQ}(2, \mathcal{N})$ are free to rotate without changing diameters of the regions and without changing the colatitudes of the collars.

Here we complete the specification of the partition algorithm by specifying a $SO(2)$ rotation for each collar which aims to maximize the minimum distance between the codepoints of successive collars of $\mathrm{EQ}(2, \mathcal{N})$.

The offset rotation applied to collar $i+1$ of $\mathrm{EQ}(2,\mathcal{N})$ with respect to collar i is

$$\text{offset} \;=\frac{\pi}{\mathsf{m}_{i+1}}-\frac{\pi}{\mathsf{m}_{i}}+\frac{\pi\,\mathrm{gcd}(\mathsf{m}_{i},\mathsf{m}_{i+1})}{\mathsf{m}_{i}\;\;\mathsf{m}_{i+1}}. \tag{4.1.1}$$

The first two terms above align the first codepoint in collar $i+1$ directly south of the first codepoint in collar i.

The third term adds an extra rotation which maximizes the minimum difference in longitude between points of collar i and points of collar $i+1$. This is because the greatest common divisor $g := \mathrm{gcd}(\mathsf{m}_i, \mathsf{m}_{i+1})$ is the smallest positive integer such that the equation

$$g = x\;\;\mathsf{m}_i + y\;\;\mathsf{m}_{i+1} \tag{4.1.2}$$

has a solution in integers. This characterization of g is a well-known result in number theory. See, for example, [114, Theorem 1.4, pp. 4–5].

As a result of (4.1.2) we have

$$\frac{g}{\mathsf{m}_i\mathsf{m}_{i+1}}=\frac{y}{\mathsf{m}_i}+\frac{x}{\mathsf{m}_{i+1}}$$

for some integer x, y, implying that since the first codepoint of collar $i+1$ is aligned with the first codepoint of collar $i+1$, there must be some codepoint of collar $i+1$ which differs in longitude by $2\pi\; g/(\mathsf{m}_i\mathsf{m}_{i+1})$ from some codepoint of collar i, and that this is the smallest non-zero difference in longitude.

4.2 The EQ codes are not good for polynomial interpolation

Here we elaborate the point made in Chapter 1, that the EQ codes for square numbers of points of $\mathbb{S}^2$, are not suitable for polynomial interpolation, because the corresponding Gram matrix is often singular to machine precision. In terms of Definition 2.10.1, the EQ codes for square numbers of points of $\mathbb{S}^2$ are either not fundamental systems or are very close to a code which is not a fundamental system.

It can in fact be proven that the stronger condition holds asymptotically in degree for $d > 1$.

Theorem 4.2.1. *For $d > 1$ there is a polynomial degree t_0, depending on d, such that for $t > t_0$, there is a non zero polynomial of degree t which is zero at all codepoints of the spherical code $\mathrm{EQP}(d, \mathcal{D}(d,t))$.*

Theorem 4.2.1 is true essentially because for large polynomial degrees the corresponding EQ codes are concentrated on too few colatitudes to ensure that the interpolating is unique. Details are given in the proof below.

For at least $d = 2$ there is strong numerical evidence for the following conjecture.

Conjecture 4.2.2. *For $d > 1, t > 0$ the set $\mathrm{EQP}(d, \mathcal{D}(d,t))$ is not a fundamental system.*

This conjecture might conceivably be approached using methods similar to those of zu Castell, Laín Fernández and Xu [166, 169] [91, Section 2.4] [92, Section 2].

Proof of Theorem 4.2.1.

We need in general to show that there is a non zero polynomial of degree t which is zero at all codepoints of the spherical code $\mathrm{EQP}(d, \mathcal{D}(d,t))$. This is easy to do if the number of zones $\mathsf{n} + 2$ is less than $t + 1$, since in this case we can construct a polynomial in the single variable x_{d+1} with zeros at the $\mathsf{n} + 2$ colatitudes of the codepoints of $\mathrm{EQP}(d, \mathcal{D}(d,t))$. We must therefore show that there exists a strength t_0 such that for $t > t_0$ the partition $\mathrm{EQ}(d, \mathcal{D}(d,t))$ has $\mathsf{n} < t - 1$.

We therefore examine $\mathrm{EQ}(d, \mathcal{N})$ for $\mathcal{N} = \mathcal{D}(d,t)$. Using (2.6.6) we have

$$\mathcal{N} = \mathcal{D}(d,t) = \frac{2t+d}{d!}\ (t+1)_{d-1} \leqslant \frac{2(t+d-1)^d}{d!}. \tag{4.2.1}$$

We now assume that $\mathcal{N} > \mathcal{N}_0\left(\frac{1}{2}\right)$ and use the definition (2.3.35) and the estimates (3.5.50) and (3.5.51) which together imply that

$$\begin{aligned}\mathsf{n}^d &\leqslant \frac{\pi^d}{\delta_I^d} = \frac{\pi^d}{\omega_d}\,\mathcal{N} = \frac{\pi^d}{\omega_d}\,\mathcal{D}(d,t)\\ &\leqslant \pi^d\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{2\,\pi^{\frac{d+1}{2}}}\,\frac{2(t+d-1)^d}{\Gamma(d+1)} \leqslant \frac{\pi^d}{\pi^{\frac{d+1}{2}}}\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d+1)}\,(t+d-1)^d.\end{aligned} \tag{4.2.2}$$

We now apply the estimate (2.2.13) to obtain

$$\mathsf{n}^d \leqslant \left(\frac{\pi}{4}\right)^{\frac{d-1}{2}}\,(t+d-1)^d.$$

If we let

$$\mathsf{C}_d := \left(\frac{\pi}{4}\right)^{\frac{d-1}{2d}}, \tag{4.2.3}$$

then for $\mathcal{N} > \mathcal{N}_0\left(\frac{1}{2}\right)$ and

$$t > \frac{(d-1)\ \mathsf{C}_d + 1}{1 - \mathsf{C}_d}$$

we have $\mathsf{n} \leqslant t-1$. $\square$

4.3 Minimum distance between codepoints

By the minimum distance between codepoints of $\mathrm{EQP}(d,\mathcal{N})$ we mean the minimum Euclidean distance, defined as follows.

Definition 4.3.1.

$$\operatorname{mindist}(d,\mathcal{N}) := \operatorname{mindist}\ \mathrm{EQP}(d,\mathcal{N}) = \min_{\mathbf{a},\mathbf{b}\in\mathrm{EQP}(d,\mathcal{N}),\mathbf{a}\neq\mathbf{b}} \|\mathbf{a}-\mathbf{b}\|\,. \tag{4.3.1}$$

The codepoints of an EQ point set are well separated in the following natural sense.

Theorem 4.3.2. *For each $d \geqslant 1$ there is a constant K''_d such that for all $\mathcal{N} \geqslant 2$, $\mathrm{EQP}(d,\mathcal{N})$ has no two codepoints with Euclidean distance less than $K''_d\ \mathcal{N}^{-\frac{1}{d}}$. In other words,*

$$\mathrm{mindist}(d,\mathcal{N}) \geqslant K''_d\ \mathcal{N}^{-\frac{1}{d}}. \tag{4.3.2}$$

The proof of Theorem 4.3.2 first shows that the minimum spherical distance between codepoints must be at least twice the minimum spherical distance between any codepoint and the boundary of the region which contains the codepoint. This implies that the minimum Euclidean distance between codepoints must be at least twice the sine of the minimum spherical distance between any codepoint and the boundary of the region which contains the codepoint. It is therefore useful to define the following quantity.

Definition 4.3.3.

$$\mathrm{minsin}(d,\mathcal{N}) := \min_{\mathsf{R} \in \mathrm{EQ}(d,\mathcal{N})} \sin \mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}). \tag{4.3.3}$$

The proof of Theorem 4.3.2 continues by using the following result.

Lemma 4.3.4. *For each $d \geqslant 1$ there is a constant K'_d such that for all $\mathcal{N} \geqslant 2$, the minimum of the sine of the spherical distance between any codepoint and the boundary of the region which contains the codepoint is at least $K'_d\ \mathcal{N}^{-\frac{1}{d}}$. In other words,*

$$\mathrm{minsin}(d,\mathcal{N}) \geqslant K'_d\ \mathcal{N}^{-\frac{1}{d}}. \tag{4.3.4}$$

Proof of Theorem 4.3.2.

We consider the general case where $d \geqslant 1$ and $\mathcal{N} > 1$.

Consider any two regions A, B of $\mathrm{EQ}(d,\mathcal{N})$ and their corresponding codepoints $\mathbf{a} := \boxdot\, A$ and $\mathbf{b} := \boxdot\, B$. The minimal geodesic arc from $\mathbf{a}$ to $\mathbf{b}$ must pass through ∂A, the boundary of A, may possibly pass through other regions and must then pass through ∂B, the boundary of B. The spherical distance $\mathsf{s}(\mathbf{a}, \mathbf{b})$ must therefore be at least the twice the minimum of $\mathsf{s}(\mathbf{a}, \partial A)$ and $\mathsf{s}(\mathbf{b}, \partial B)$.

Since our argument works for any two codepoints, the minimum spherical distance between codepoints must be at least twice the minimum spherical distance between any codepoint and the boundary of the region which contains the codepoint. In other words,

$$\min_{\mathbf{a},\mathbf{b} \in \mathrm{EQP}(d,\mathcal{N}), \mathbf{a} \neq \mathbf{b}} \mathsf{s}(\mathbf{a}, \mathbf{b}) \geqslant 2 \min_{\mathsf{R} \in \mathrm{EQP}(d,\mathcal{N})} \mathsf{s}(\boxdot\,\mathsf{R}, \partial \mathsf{R}).$$

We therefore have

$$\begin{aligned}
\mathrm{mindist}(d,\mathcal{N}) &= \min_{\mathbf{a},\mathbf{b} \in \mathrm{EQP}(d,\mathcal{N}), \mathbf{a} \neq \mathbf{b}} \Upsilon\left(\mathsf{s}(\mathbf{a}, \mathbf{b})\right) = \min_{\mathbf{a},\mathbf{b} \in \mathrm{EQP}(d,\mathcal{N}), \mathbf{a} \neq \mathbf{b}} 2\ \sin \frac{\mathsf{s}(\mathbf{a}, \mathbf{b})}{2} \\
&= 2\ \sin \frac{\min_{\mathbf{a},\mathbf{b} \in \mathrm{EQP}(d,\mathcal{N}), \mathbf{a} \neq \mathbf{b}} \mathsf{s}(\mathbf{a}, \mathbf{b})}{2} \\
&\geqslant 2\ \sin \min_{\mathsf{R} \in \mathrm{EQP}(d,\mathcal{N})} \mathsf{s}(\boxdot\,\mathsf{R}, \partial \mathsf{R}) = 2 \min_{\mathsf{R} \in \mathrm{EQP}(d,\mathcal{N})} \sin \mathsf{s}(\boxdot\,\mathsf{R}, \partial \mathsf{R}) \\
&= 2\ \mathrm{minsin}(d,\mathcal{N}).
\end{aligned}$$

Using Lemma 4.3.4, we see that there is a constant K'_d such that for all $\mathcal{N} \geqslant 2$, we have

$$\mathrm{mindist}(d,\mathcal{N}) \geqslant 2\ \mathrm{minsin}(d,\mathcal{N}) \geqslant 2\ K'_d\ \mathcal{N}^{-\frac{1}{d}}.$$

□

4.3.1 Sketch of proof of Lemma 4.3.4

The proof of Lemma 4.3.4 proceeds by induction, with the unit circle as a special case. In the case of the unit circle, the codepoints are equally spaced and we show that the distance between each codepoint and the corresponding boundary has the right order.

For $d > 1$ we show that there is a trivial lower bound of the right order for small $\mathcal{N}$, since the no codepoint lies on the boundary of a region.

We now consider the general case where $d > 1$ and $\mathcal{N} > 1$. We make the inductive assumption that there is a constant $K_{d-1}^{'}$ such that for all $m > 1$,

$$\operatorname{minsin}(d-1, m) \geqslant K_{d-1}^{'}\ m^{\frac{1}{1-d}}. \tag{4.3.5}$$

If a given region R is a polar cap then the codepoint $\boxdot\,\mathsf{R}$ is a pole and $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R})$, the spherical distance between $\boxdot\,\mathsf{R}$ and the boundary of R, is ϑ_c, the spherical radius of the polar cap. We show that $\sin\vartheta_c$ is of the correct order.

If R is not a polar cap then it is a region contained in a collar, say collar i. In this case the point of $\partial\mathsf{R}$ closest to the codepoint $\boxdot\,\mathsf{R}$ is either a point of the top and bottom boundary $\partial^{\updownarrow}\mathsf{R}$ or a point of the side boundary $\partial_{\leftrightarrow}\mathsf{R}$.

We show that $\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$ is half the collar angle δ_i. We also show that when $\sin\mathsf{s}(\boxdot\,\mathsf{R}, \partial_{\leftrightarrow}\mathsf{R}) < \sin\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$ we have

$$\sin\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) = \sin\frac{\vartheta_i + \vartheta_{i+1}}{2}\ \ \sin\mathsf{s}(\boxdot\,\Pi\,\mathsf{R}, \partial\,\Pi\,\mathsf{R}).$$

Using our inductive assumption we deduce that

$$\sin\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) \geqslant \min\left(\sin\frac{\delta_i}{2}, K_{d-1}^{'}\sin\frac{\vartheta_i + \vartheta_{i+1}}{2}\ \ \mathsf{m}_i^{\frac{1}{1-d}}.\right).$$

We now assume that $\mathcal{N} \geqslant N_0$ with N_0 sufficiently large that we have at least five collars. We then use the definitions and estimates which were used in the proof of the lemmas supporting Theorem 3.1.3 to show that $\sin\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R})$ is of the correct order.

4.3.2 Analysis of the case $d > 1$ and $\mathcal{N} > 2$

We now analyze $\operatorname{minsin}(d, \mathcal{N})$ in the case $d > 1$ and $\mathcal{N} > 2$ in order to develop a number of geometrical lemmas which will be used in the proof of Lemma 4.3.4.

Since $\mathcal{N} > 2$ we have at least one collar. In this case, a region is a polar cap, is the only region in a collar, or is one of many in a collar.

If the codepoint $\boxdot\,\mathsf{R}$ is a pole contained in a polar cap R, then $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) = \vartheta_c$, the spherical radius of the polar cap. Otherwise the codepoint is in a region contained in a collar.

We now assume that we have a codepoint $\boxdot\,\mathsf{R}$ which is the centre point of a region R contained in collar i. Denote the top and bottom boundary of R as $\partial^{\updownarrow}\mathsf{R}$. Denote the side boundary of R as $\partial_{\leftrightarrow}\mathsf{R}$.

If R is the only region in a collar, then $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R})$ is $\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$, the distance between $\boxdot\,\mathsf{R}$ and $\partial^{\updownarrow}\mathsf{R}$, the top and bottom boundaries of R.

If R is one of many in a collar then $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R})$ is the minimum of $\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$ and $\mathsf{s}(\boxdot\,\mathsf{R}, \partial_{\leftrightarrow}\mathsf{R})$, the distance between $\boxdot\,\mathsf{R}$ and the side boundary of R.

We now consider $\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$.

Lemma 4.3.5. *Let R be a region in collar i of $\mathrm{EQ}(d, \mathcal{N})$. Then the distance from the codepoint $\boxdot\,\mathsf{R}$ to the top and bottom boundary $\partial^{\updownarrow}\mathsf{R}$ is*

$$\mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R}) = \frac{\delta_i}{2}. \tag{4.3.6}$$

The analysis of $\mathsf{s}(\boxdot\,\mathsf{R}, \partial_{\leftrightarrow}\mathsf{R})$ is more complicated, but it gives us the basis of an inductive proof of Lemma 4.3.4.

Lemma 4.3.6. *For a region R in collar i of $\mathrm{EQ}(d, \mathcal{N})$ such that $\mathsf{m}_i \geqslant 2$, either $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) = \mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R}) < \mathsf{s}(\boxdot\,\mathsf{R}, \partial_{\leftrightarrow}\mathsf{R})$ or $\mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) = \mathsf{s}(\boxdot\,\mathsf{R}, \partial_{\leftrightarrow}\mathsf{R}) \leqslant \mathsf{s}(\boxdot\,\mathsf{R}, \partial^{\updownarrow}\mathsf{R})$ and*

$$\sin \mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) = \sin\frac{\vartheta_i + \vartheta_{i+1}}{2}\ \sin \mathsf{s}(\boxdot\,\Pi\,\mathsf{R}, \partial\,\Pi\,\mathsf{R}). \tag{4.3.7}$$

Using the inductive assumption that for $\mathsf{R} \in \mathrm{EQ}(d-1, \mathsf{m}_i)$ we have

$$\sin \mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) \geqslant K'_{d-1}\mathsf{m}_i^{\frac{1}{1-d}},$$

we conclude that

$$\sin \mathsf{s}(\boxdot\,\mathsf{R}, \partial\mathsf{R}) \geqslant \min\left(\sin\frac{\delta_i}{2}, K'_{d-1}\sin\frac{\vartheta_i + \vartheta_{i+1}}{2}\ \mathsf{m}_i^{\frac{1}{1-d}}\right).$$

We now define

$$\psi_i := \sin \frac{\vartheta_i + \vartheta_{i+1}}{2} \ \mathsf{m}_i^{\frac{1}{1-d}}. \tag{4.3.8}$$

We therefore have

$$\operatorname{minsin}(d, \mathcal{N}) \geqslant \min \left(\sin \vartheta_c, \min_{i=1}^{\mathcal{N}} \sin \frac{\delta_i}{2}, K_{d-1}^{'} \min_{i=1}^{\mathcal{N}} \psi_i \right). \tag{4.3.9}$$

We now assume that $\mathcal{N} \geqslant N_0$ with N_0 sufficiently large that we have at least five collars. We can therefore use the definitions and estimates of the previous chapter. We use the feasible domains $\mathbb{D}, \mathbb{D}_+, \mathbb{D}_t$ and $\mathbb{D}_{m+}$ defined by (3.5.10), (3.5.42), (3.5.38) and (3.5.44) respectively, and the functions $\mathcal{Y}, \mathcal{T}, \mathcal{B}, \mathcal{M}$ and Δ, defined by (3.5.20) to (3.5.24) respectively.

We also define

$$\Psi(\tau, \beta, \vartheta) := \sin \frac{\mathcal{T}(\tau, \vartheta) + \mathcal{B}(\beta, \vartheta)}{2} \ \mathcal{M}(\tau, \beta, \vartheta)^{\frac{1}{1-d}}, \tag{4.3.10}$$

where $\mathcal{T}, \mathcal{B}$ and $\mathcal{M}$ are defined by (3.5.21), (3.5.22) and (3.5.24) respectively.

The sequence ψ and the function Ψ are related to each other in the same way as the sequence δ and the function Δ. In other words, we have the following relationship.

Lemma 4.3.7. *For $i \in \{1, \ldots, \mathsf{n}\}$ we have*

$$\Psi(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) = \psi_i. \tag{4.3.11}$$

The function Ψ also has the same symmetry as the function Δ.

Lemma 4.3.8. *The function Ψ satisfies*

$$\Psi(\tau, \beta, \pi - \vartheta) = \Psi(\beta, \tau, \vartheta - \delta_F). \tag{4.3.12}$$

As a consequence of (3.5.31), (4.3.9) and (4.3.11) we have

$$\operatorname{minsin}(d,\mathcal{N}) \geqslant \min\left(\sin\vartheta_c, \min_{\mathbb{D}} \sin\frac{\Delta}{2}, K'_{d-1} \min_{\mathbb{D}} \Psi\right), \tag{4.3.13}$$

and by Lemmas 3.5.9 and 4.3.8 we therefore have

$$\operatorname{minsin}(d,\mathcal{N}) \geqslant \min\left(\sin\vartheta_c, \min_{\mathbb{D}_+} \sin\frac{\Delta}{2}, K'_{d-1} \min_{\mathbb{D}_+} \Psi\right). \tag{4.3.14}$$

To complete the proof of Lemma 4.3.4 we must show that each of the expressions in (4.3.14) has a lower bound of the correct order.

Lemma 4.3.9. *For $d > 1$, there is a positive constant $N'_c \in \mathbb{N}$ and a monotonic increasing positive real function K'_c such that for each partition $\mathrm{EQ}(d,\mathcal{N})$ with $\mathcal{N} > x \geqslant N'_c$,*

$$\sin\vartheta_c \geqslant K'_c(x)\mathcal{N}^{-\frac{1}{d}}.$$

Lemma 4.3.10. *For $d > 1$, there is a positive constant $N'_\Delta \in \mathbb{N}$ and a monotonic increasing positive real function K'_Δ such that for each partition $\mathrm{EQ}(d,\mathcal{N})$ with $\mathcal{N} > x \geqslant N'_\Delta$,*

$$\min_{\mathbb{D}_+} \sin\frac{\Delta}{2} \geqslant K'_\Delta(x)\mathcal{N}^{-\frac{1}{d}}.$$

Lemma 4.3.11. *For $d > 1$, there is a positive constant $N_\Psi \in \mathbb{N}$ and a monotonic increasing positive real function C'_Ψ such that for each partition $\mathrm{EQ}(d,\mathcal{N})$ with $\mathcal{N} > x \geqslant N_\Psi$,*

$$\min_{\mathbb{D}_+} \Psi \geqslant C'_\Psi(x)\mathcal{N}^{-\frac{1}{d}}.$$

4.3.3 Numerical results

Figures 4.2, 4.3 and 4.4 show the minimum distance coefficient for $\mathrm{EQP}(d,\mathcal{N})$ for $\mathcal{N}$ from 2 to 20000 for $d = 2, 3, 4$ respectively. The minimum distance coefficient is here defined to be $\text{mindist}(d, \mathcal{N})\mathcal{N}^{\frac{1}{d}}$, where $\text{mindist}(d, \mathcal{N})$ is defined by Definition 4.3.1.

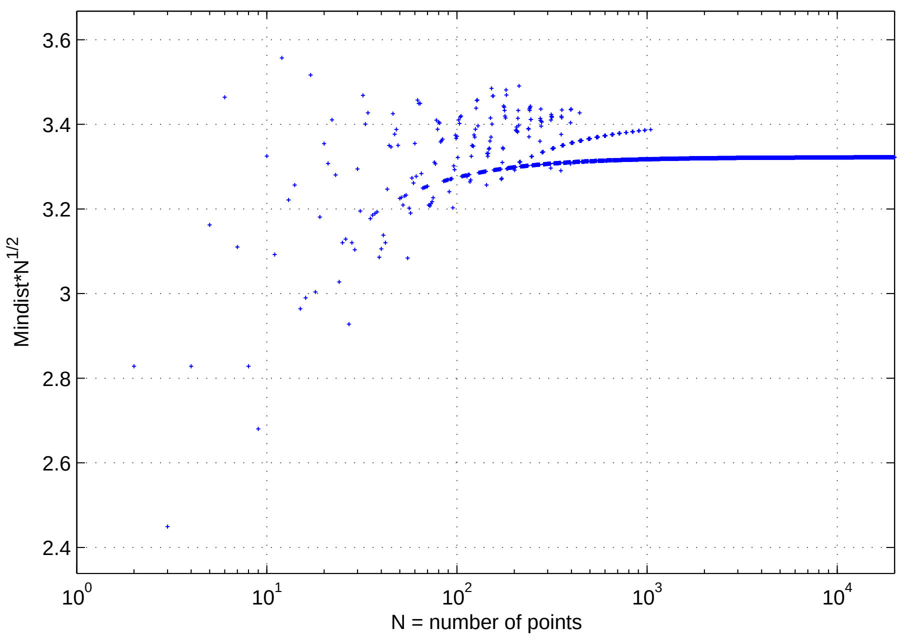

Figure 4.2: Minimum distance coefficient of $\text{EQP}(2, \mathcal{N})$ (semi-log scale)

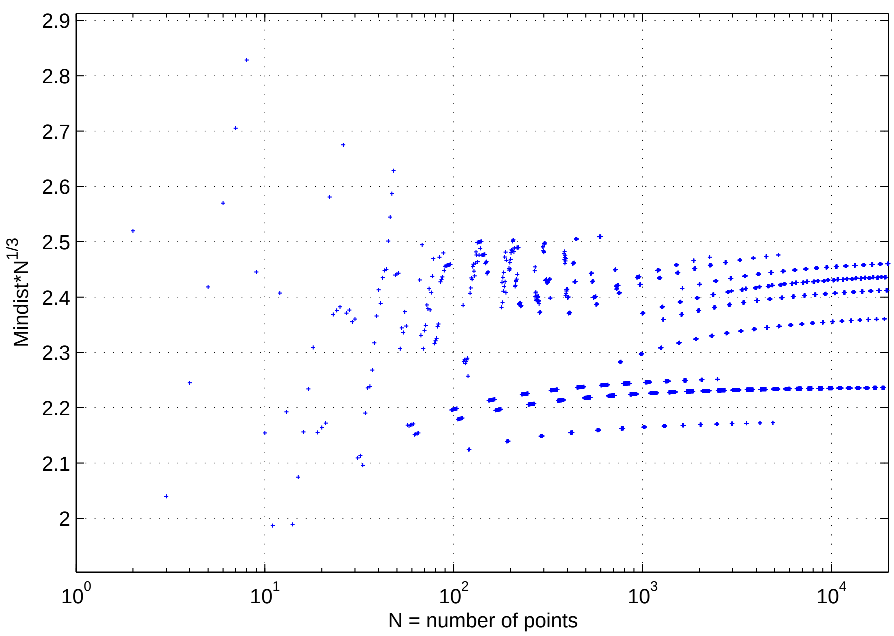

Figure 4.3: Minimum distance coefficient of $\text{EQP}(3, \mathcal{N})$ (semi-log scale)

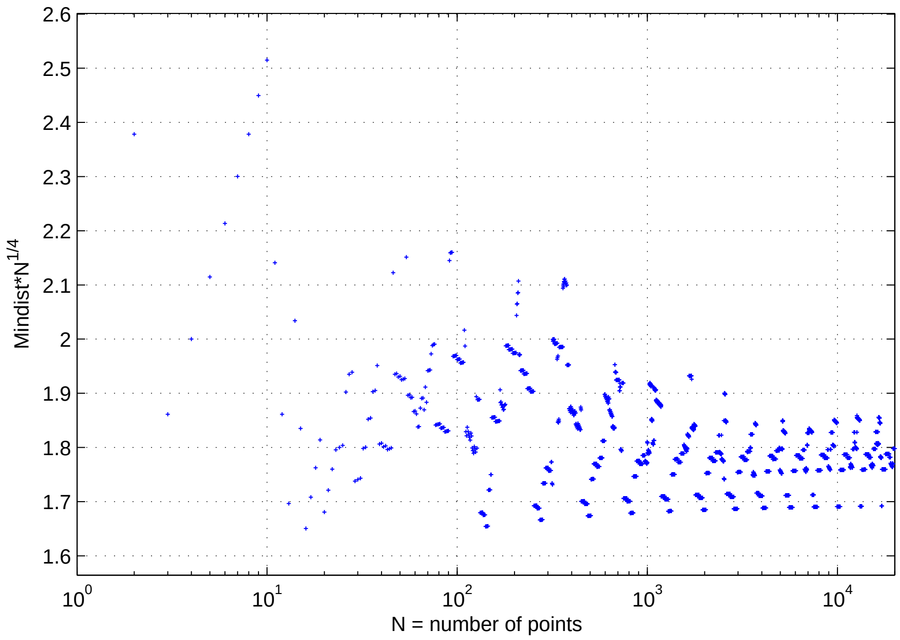

Figure 4.4: Minimum distance coefficient of EQP$(4, \mathcal{N})$ (semi-log scale)

4.4 Packing density

Hamkins, in his PhD thesis [65], and Hamkins and Zeger, in a series of papers, describe "asymptotically optimal" spherical codes. According to [65, p. 64], [66] their wrapped spherical codes for $\mathbb{S}^d$ are in general, asymptotically optimal in terms of packing density, if the densest packing in $\mathbb{R}^d$ is used in the wrapping algorithm. Their laminated codes [67], have in some cases a higher packing density than the corresponding wrapped spherical codes [65, p. 80].

The packing density of the EQ points is in general lower than that of the Hamkins-Zeger wrapped spherical codes. Table 4.1 compares the maximum number of points of the Hamkins-Zeger spherical codes and EQP for selected values of d and minimum distance. Compare this with Tables 3.2 and 3.3 of [65, p. 47].

For $\mathbb{S}^2$, taking the rotation of Section 4.1.2 into account, the codepoints of EQP$(2, \mathcal{N})$ near the equator approximate a non-uniform hexagonal lattice as $\mathcal{N} \to \infty$. Without this rotation, the codepoints near the equator approximate a square lattice of distance $\sqrt{\omega_d / \mathcal{N}}$ as $\mathcal{N} \to \infty$.

d	2	2	3
Minimum distance	0.1	0.01	0.1
Coxeter upper bound	1450	145 103	29 364
Laminated code	1294	124 422	16 976
Wrapped code	1070	130 682	17 198
EQP	1100	110 366	13 591

Table 4.1: Maximum number of points for given minimum distance for Hamkins-Zeger spherical codes and EQP.

In general, for $d > 2$ we should not expect $\mathrm{EQP}(d, \mathcal{N})$ to yield a better packing density than the density of the simple cubic lattice for $\mathbb{R}^d$. This density is simply the ratio of the volume of the unit ball $\mathbb{B}^d$ to the volume of the enclosing cube, 2^d, that is

$$\text{density}\left(\mathrm{EQP}(d, \mathcal{N})\right) \approx \frac{\mu(\mathbb{B}^d)}{2^d} = \frac{\pi^{d/2}}{2^d\ \Gamma(d/2+1)}. \tag{4.4.1}$$

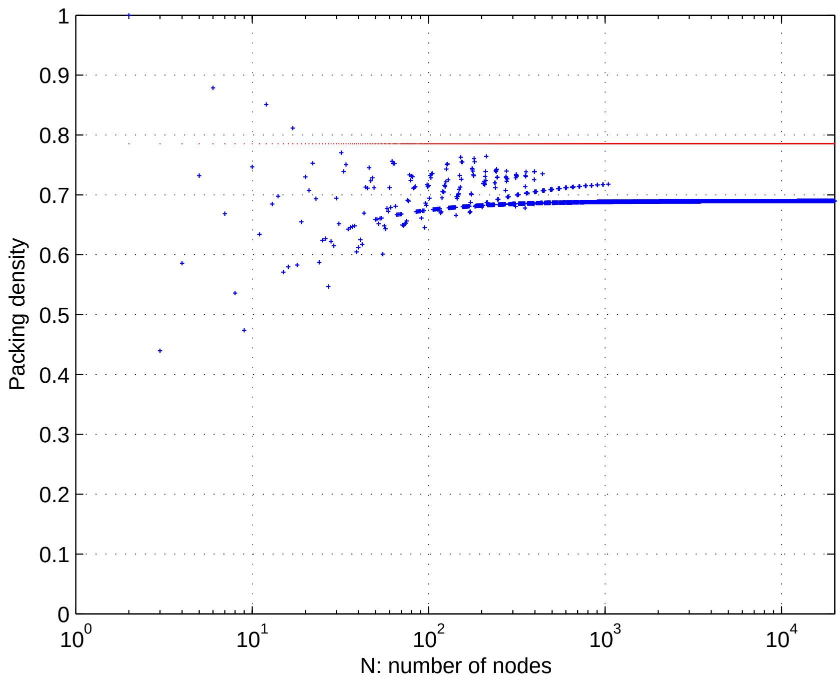

Figure 4.5: Packing density of $\mathrm{EQP}(2, \mathcal{N})$ (semi-log scale)

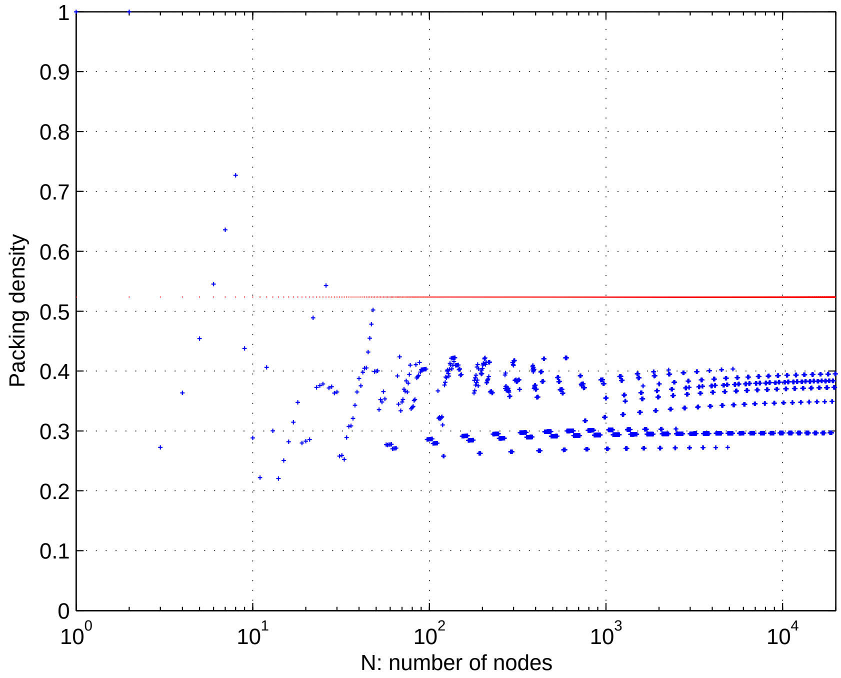

Figure 4.6: Packing density of EQP$(3, \mathcal{N})$ (semi-log scale)

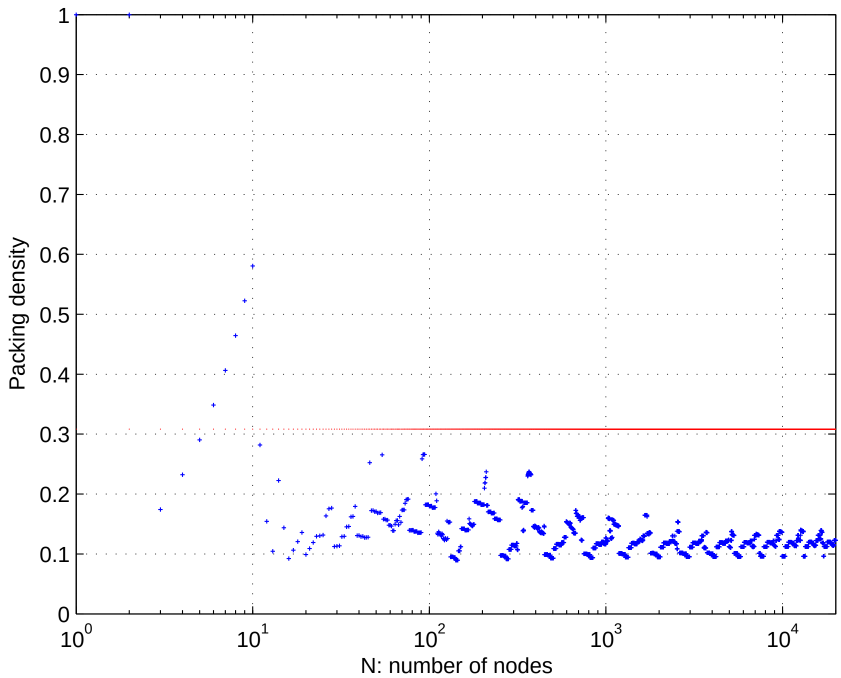

Figure 4.7: Packing density of EQP$(4, \mathcal{N})$ (semi-log scale)

Figures 4.5, 4.6 and 4.7 show the packing density of $\mathrm{EQP}(d,\mathcal{N})$ for $\mathcal{N}$ from 2 to 20000 for $d = 2,3,4$ respectively. In each of these figures the red horizontal line is the the density of the simple cubic lattice for $\mathbb{R}^d$, as per (4.4.1).

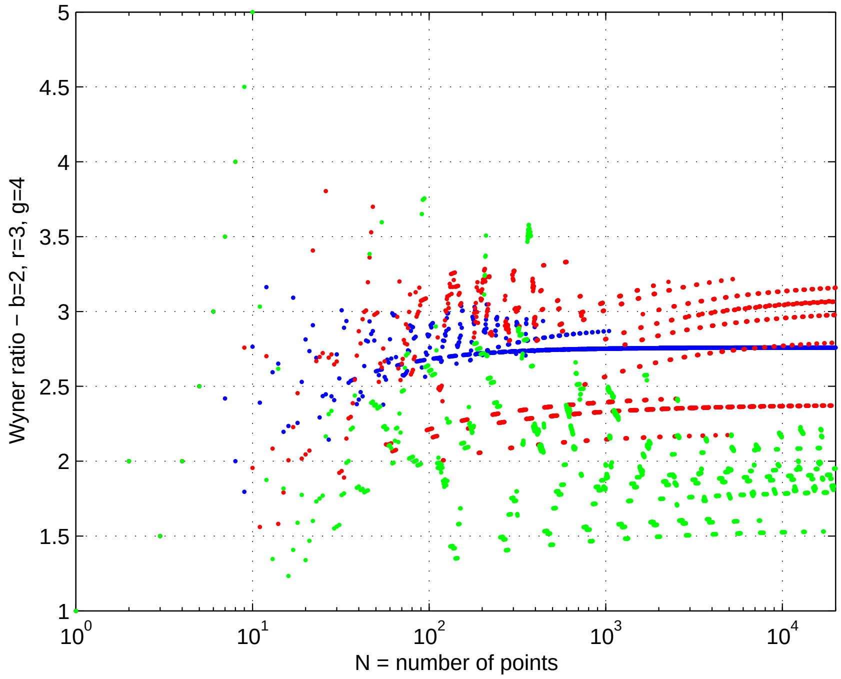

Figure 4.8: Wyner ratios for EQP(2), EQP(3), EQP(4) (semi-log scale)

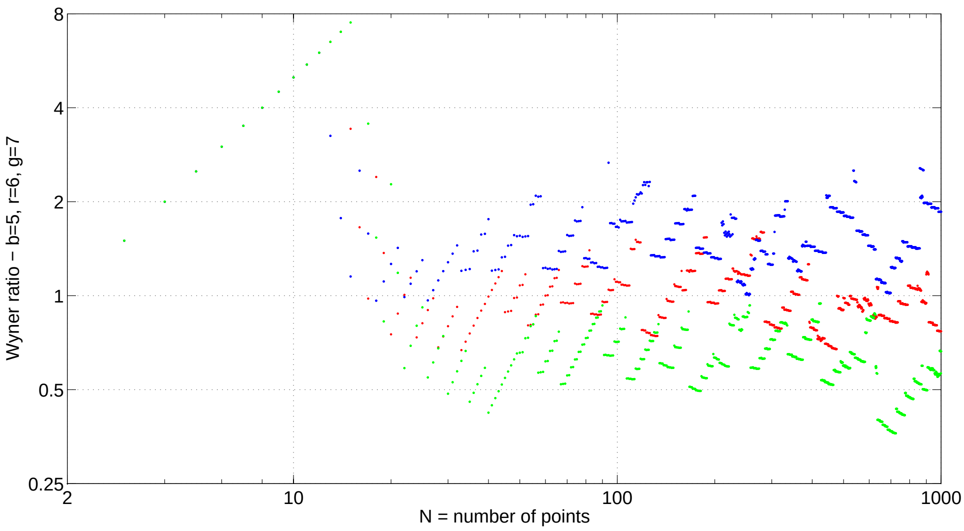

Figure 4.9: Wyner ratios for EQP(5), EQP(6), EQP(7) (log-log scale)

We now examine the packing density of $\text{EQP}(d, \mathcal{N})$ using the Wyner ratio as per Definition 2.7.6. Figure 4.8 shows the Wyner ratio of $\text{EQP}(d, \mathcal{N})$ for $\mathcal{N}$ from 2 to 20000 for $d = 2$ in blue, $d = 3$ in red and $d = 4$ in green. Each shows a Wyner ratio greater than 1, but $d = 2$ is better than $d = 3$ which is better than $d = 4$.

Figure 4.9 shows the Wyner ratio of $\text{EQP}(d, \mathcal{N})$ for $\mathcal{N}$ from 2 to 1000 for $d = 5$ in blue, $d = 6$ in red and $d = 7$ in green. For $d = 5$ the Wyner ratio is generally better than 1, but not so for $d = 6$ and especially not for $d = 7$. For $d = 7$ the Wyner ratio is generally worse than 1, for $\mathcal{N} > 20$. This indicates that $\text{EQP}(d, \mathcal{N})$ yields a poor packing for $d > 5$.

4.4.1 Nesting and layering

The construction of the EQ codes can be modified to produce nested codes, by subdividing each region of an EQ partition into 3^{nd} regions, where n is the nesting depth. We elaborate this idea here. This is an informal discussion containing no proofs.

The main idea here is to divide each collar into 3^n collars of equal area. To simplify the discussion, we only consider $n = 1$ here. For $n > 1$ we can apply the same method recursively. We first divide collar i of $\text{EQ}(d, \mathcal{N})$ into 3 collars of equal area. At this point each of these 3 collars is still subdivided into m_i regions. We then recursively apply the subdivision to $\text{EQ}(d-1, \mathsf{m}_i)$ to subdivide each of the collars into $3^{d-1}\ \mathsf{m}_i$ regions. We need to use a different algorithm for the spherical caps, where we must subdivide the cap into a smaller cap having 3^{-d} of the original area, and two new collars.

The nested code is obtained by placing codepoints at the centres of each region, except where the region contains an original codepoint, which is retained.

One difficulty with this method of subdivision is that the definition of the centre of a region given in Section 4.1.1 simply divides a collar by halving the colatitude. Since $D\mathcal{V}(\vartheta)$ is proportional to $\sin^{d-1}\theta$ this results in the half of the collar lying towards the equator having more area than the half lying towards the poles, with the problem getting worse as d increases. There is therefore a value of d such that our simple subdivision scheme no longer works, in the sense that when we subdivide a collar into 3 new collars, the colatitude of the original codepoints will no longer lie in the centre collar of the 3 new collars, but instead will lie in the new collar which is closest to the pole.

The remedy to this difficulty is to revise the definition of the centre point of a region so that each collar is divided into two by area rather than by colatitude. With this new definition of centre point, the subdivision process will retain the original codepoints as the centre points of $\mathcal{N}$ of the $3^d\ \mathcal{N}$ new regions.

4.4.2 Methods to increase density

The EQ partition of $\mathbb{S}^2$ has its regions offset in a natural way to maximize minimum distance between the corresponding EQ codepoints. For $\mathbb{S}^d$ with $d > 2$ there is no such natural offset, but it is possible that a variation of the EQ algorithm could produce a larger minimum distance.

To elaborate, we have already seen in Section 4.1.2 a scheme to maximize the minimum distance between codepoints by rotating successive collars. For $d \geqslant 3$, if the EQ partition contains more than one collar then each collar contains a codepoint of the EQ code corresponding to the North Pole of $\mathbb{S}^{d-1}$, and the minimum distance between codepoints of collar i and collar $i+1$ is $\frac{\delta_i+\delta_{i+1}}{2}$. Any $SO(d)$ rotation which does not fix the North Pole could be used to move the “North Pole” of collar $i+1$ away from the “North Pole” of collar i, but this causes two further problems.

First, unlike the $\mathbb{S}^2$ case, it is not clear which rotation will maximize the minimum distance between codepoints of successive collars. Second, a general $SO(d)$ rotation will not take a RISC region to a RISC region: each pseudovertex will in general have its colatitudes perturbed in a different way. For $d = 3$, the Matlab implementation EQSP 1.10 “solves” the first problem by ignoring it. It rotates collar $i+1$ by an $SO(3)$ rotation which takes the North Pole of $\mathbb{S}^2$ to a point where the boundary of the North polar cap of collar i meets the boundary between two adjacent regions in the top collar of the EQ partition of collar i. EQSP 1.10 solves the second problem by recording the rotation used for each collar. To describe a region of its modified version of $\mathrm{EQ}(3,\mathcal{N})$ requires the coordinates of two pseudovertices in standard position as well as a 3×3 matrix describing the rotation of the collar. This method is clumsy and was not attempted for $d > 3$.

There is an alternative to rotating the collars, but it involves a modified partition algorithm. The standard EQ algorithm forces the creation of a cap at both the North and South poles. The modified "*unicap*" EQ algorithm forces a cap only at one pole, dividing the remainder of the sphere as per the collars of the standard EQ algorithm. In some cases, this may still result in a cap at the opposite pole, but in most cases in low dimensions, it will result in the opposite pole not having a cap. Also, the unicap EQ partition may result in a lager maximum diameter than the standard EQ algorithm, or the corresponding unicap EQ codes may have a smaller minimum distance than the standard EQ codes.

The idea of the modified EQ partition is to use the unicap EQ partition for the collars, with the polar caps alternating between the North and South poles. This may result in a larger minimum distance between codepoints of the corresponding modified EQ code. This idea has not yet been tried.

4.4.3 Combined nesting and rotation

In the case of the $\mathbb{S}^2$ partitions, the nesting scheme of Section 4.4.1 lends itself to improvement by $SO(2)$ rotations of the new collars.

Consider the case where each collar is split into three new collars. For a single original collar, consider the three new collars and number them $3i$ (the upper collar), $3i + 1$ (the middle collar) and $3i + 2$ (the lower collar). Since we want to preserve the original codepoints, we can't rotate the middle collar, but we can rotate the upper and lower collars by half the difference in longitude between codepoints of the middle collar. This is an optimal arrangement of codepoints of the 3 new collars, but it disturbs the relationship between collar $3i$ and collar $3i - 1$ (the lower collar of the adjacent group of three collars). The solution would be to treat collars $3i - 2$ and $3i + 1$ as fixed and to jointly optimize the rotations of collars $3i - 1$ and $3i$. This has not yet been tried.

4.5 Spherical coding and decoding algorithms

The EQ partitions and EQ codes can be used to define a scheme for spherical coding and decoding and a related scheme for Gaussian source coding, similar to those described by Hamkins and Zeger [66, 68].

Refer to Figure 4.10 while reading the description of the spherical coding and decoding scheme.

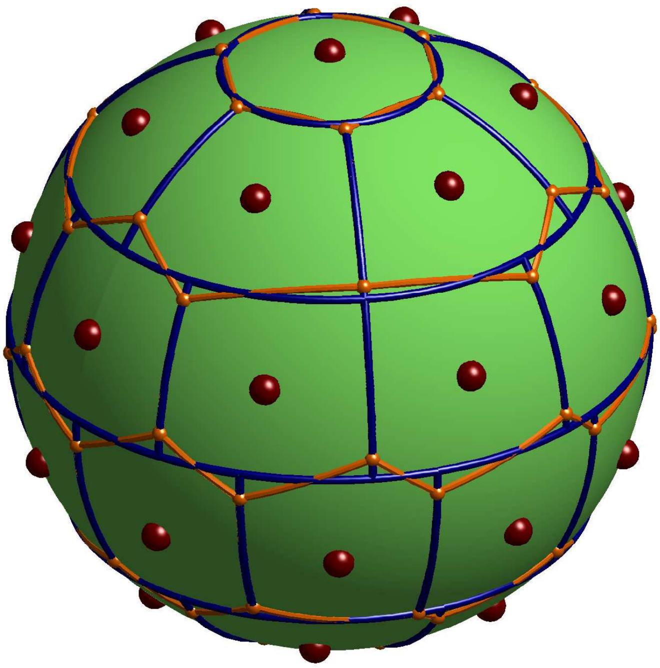

Figure 4.10: EQ code $\mathrm{EQP}(2, 33)$, Voronoi cells, and $\mathrm{EQ}(2, 33)$

Figure 4.10 illustrates the code $\mathrm{EQP}(2, 33)$ in red, with the Voronoi cells shown in yellow and the boundaries of the partition $\mathrm{EQ}(2, 33)$ shown in blue. This shows that the Voronoi cells of an EQ code are not the sames as the regions of the corresponding EQ partition.

Preparation for spherical coding and decoding.

The preparation phase sets up two mappings.

1. Map a region number to a codepoint.

2. Map the pseudo-vertex coordinates of a region, expressed as the pair of d-tuples (τ, υ) to a region number.

   This mapping uses a $d{-}1$ level tree structure, using spherical polar coordinates.

   - Level 1 of the tree corresponds to the zones of the EQ partition of $\mathbb{S}^d$

     This corresponds the major colatitude, dimension d.
   - For $k \in \{2, \ldots, d-2\}$, level k of the tree corresponds to the zones of the EQ partition of $\mathbb{S}^{d-k+1}$.

     This corresponds the colatitude of dimension $d - k + 1$.
   - The leaves at level $d-1$ correspond to the zones of an EQ partition of $\mathbb{S}^2$.

     This corresponds to the colatitude of dimension 2.

     There is sufficient information at the leaves to calculate the longitude.

Spherical channel coding.

1. Map the codepoint number to codepoint coordinates.
2. Transmit the codepoint coordinates.

Spherical channel decoding - the Quasi-Nearest Codepoint algorithm.

Channel decoding takes the received coordinates and tries to determine the nearest codepoint. Since the regions are in general not the Voronoi cells of the corresponding codepoints, if we just take the received point, look up the region and then look up the corresponding codepoint, we cannot be sure that we have found the nearest codepoint to the received point. We instead use a slightly more elaborate algorithm, here called the "*Quasi-Nearest Codepoint*" algorithm:

1. Look up the received coordinates $\mathbf{x}$ to obtain the region R.

   This is done by:

   (a) Cartesian to spherical coordinate conversion:

   $\xi := (\xi_1, \ldots, \xi_d) = \odot^{-1}\,\mathbf{x}$.

   (b) For dimension i from d down to 2:

   Binary chop using colatitude ξ_i to obtain the zone number.

(c) For dimension 1:

Use the longitude ξ_1 to calculate the region number.

2. Obtain the codepoint $\mathbf{a} := \boxdot\,\mathsf{R}$.
3. Get the distance to the codepoint, $s_0 := \mathsf{s}(\mathbf{x}, \mathbf{a})$.
4. If the distance is less than the packing radius then we are done. Set the final codepoint $\mathbf{f} := \mathbf{a}$ and quit.
5. Otherwise the nearest codepoint may be the codepoint a of neighbouring region.
   - The neighbouring region cannot be one that differs only by longitude. The boundary between two regions which differ only by longitude consists of points which are equidistant from the two corresponding codepoints. We therefore concentrate on the regions which differ by colatitude.
   - Order the $d - 1$ colatitudes of ξ by increasing distance to the nearest corresponding boundary colatitude.
   - We have colatitudes ξ_{j_i}, for i from 1 to $d-1$, where j is some permutation of $\{2, \ldots, d\}$. Call the corresponding nearest boundary colatitudes ∂_{j_i}.
6. Keep the distance s_0 and codepoint $\mathbf{a}$ as the candidate distance r and candidate codepoint $\mathbf{c}$.
7. For each index i from 1 to $d - 1$:

   (a) Reflect the colatitude ξ_{j_i} into the corresponding boundary ∂_{j_i} to obtain ξ' with $\xi'_{j_i} := 2\ \partial_{j_i} - \xi_{j_i}$ and $\xi'_k := \xi_k$ for all other coordinates k.

   We have a new point $\mathbf{x}_i = \odot\,\xi'$ in a region R_i which is a neighbour of the original region.

   (b) Find the codepoint $\mathbf{a}_i = \boxdot\,\mathsf{R}_i$ of the new region.

   (c) Find the distance $s_i = \mathsf{s}(\mathbf{x}, \mathbf{a}_i)$ from the received point to the new codepoint.

   (d) If the new distance s_i is less than the packing radius we are done. Set the final codepoint $\mathbf{f} := \mathbf{a}$ and quit.

(e) Otherwise if the new distance s_i is less than distance r from the received point $\mathbf{x}$ to the candidate codepoint $\mathbf{c}$, then keep the new distance and the new codepoint as candidate distance and codepoint. In other words, set $r = s_i$, $\mathbf{c} = \mathbf{a}_i$.

8. Set the final codepoint $\mathbf{f} := \mathbf{c}$ and quit.

If $\mathsf{s}(\mathbf{x}, \mathbf{f})$ is less than the packing radius, then we are certain that $\mathbf{f}$ is the closest codepoint to $\mathbf{x}$.

In other cases, we have a conjecture, which is very plausible for $\mathbb{S}^2$ but is speculative for $\mathbb{S}^d$ for $d > 2$.

Conjecture 4.5.1. *For $d \geqslant 2$, the Quasi Nearest Codepoint algorithm is a maximum likelihood decoding algorithm for* $\mathrm{EQP}(d)$.

Remarks.

We use spherical distances in the Quasi Nearest Codepoint algorithm, but could just as easily have used Euclidean distances.

The searches used at each level of the Quasi Nearest Codepoint algorithm use binary chop, which is logarithmic in time with respect to the size of the search space. Since at each level we do a fixed number of searches depending on the dimension, it should be fairly easy to show that the Quasi Nearest Codepoint algorithm is logarithmic in time with respect to the number of codepoints. The space requirements of the Preparation step are a little harder to compute, but seem to be closer to linear with respect to the number of codepoints. For $\mathbb{S}^2$ the space required is proportional to the number of zones and is thus proportional to the square root of the number of codepoints.

In [68], Hamkins and Zeger describe Gaussian source coding and spherical quantization.

The spherical quantization scheme of Huber and Matschkal is superficially similar to the Quasi Nearest Codepoint algorithm, but there are some significant differences. The Huber-Matschkal algorithm partitions the sphere into regions which are only of similar area rather than almost exactly equal area. More precisely, looking at steps (13), (14) and (20) in [105], the regions which are created by these steps are not necessarily equal in area other than for the trivial case of $\mathbb{S}^1$. The spherical quantization algorithm described in [105] differs in two other significant ways from the Quasi Nearest Codepoint algorithm. Firstly, it is more space-efficient than the Quasi Nearest Codepoint algorithm because it does not use a codebook. Secondly, it does not try to find the nearest codepoint to a given point on the sphere, but rather merely maps the point to the region which contains it.

In [156], Utkovski and Utkovski also describe a spherical quantizer for Gaussian sources.

4.6 Proofs of Lemmas

Proof of Lemma 4.3.4.

The proof of Lemma 4.3.4 proceeds by induction, with the unit circle as a special case.

In the case of the unit circle, the $\mathcal{N}$ codepoints are equally spaced with each in the centre of its corresponding region, which is an arc of length $\frac{2\pi}{\mathcal{N}}$. Thus the distance between each codepoint and a boundary of the corresponding region is

$$\mathsf{s}(\boxdot\, \mathsf{R}, \partial \mathsf{R}) = \frac{\pi}{\mathcal{N}}.$$

We therefore have

$$\operatorname{minsin}(d, \mathcal{N}) = \sin \frac{\pi}{\mathcal{N}} \geqslant \frac{2}{\mathcal{N}} \tag{4.6.1}$$

for $\mathcal{N} \geqslant 2$.

For $d > 1$ we make the inductive assumption that there is a constant K'_{d-1} such that for all $m > 1$,

$$\mathrm{minsin}(d-1, m) \geqslant K'_{d-1}\ m^{\frac{1}{1-d}}. \tag{4.6.2}$$

For $d > 1$ there is a trivial lower bound of the right order for small $\mathcal{N}$, since the no codepoint lies on the boundary of a region. If $\mathcal{N} \leqslant N_0$, then we need only consider a finite number of partitions, each of which contain a finite number of regions. We set

$$K'_L := \min_{\mathcal{N}=2}^{N_0} \mathrm{minsin}(d, \mathcal{N})\ \mathcal{N}^{\frac{1}{d}}.$$

Then for $\mathcal{N} \in \{2, \ldots, N_0\}$, we have

$$\mathrm{minsin}(d, \mathcal{N}) \geqslant K'_L \mathcal{N}^{-\frac{1}{d}}.$$

We now consider the general case where $d > 1$ and $\mathcal{N} > 2$. The analysis of this case above results in (4.3.14) which states that

$$\mathrm{minsin}(d, \mathcal{N}) \geqslant \min\left(\sin\vartheta_c, \min_{\mathbb{D}_+} \sin\frac{\Delta}{2}, K'_{d-1} \min_{\mathbb{D}_+} \Psi\right).$$

Lemmas 4.3.9, 4.3.10 and 4.3.11 show that each of the expressions in (4.3.14) has a lower bound of the correct order. □

Proof of Lemma 4.3.5.

The top and bottom boundaries of the region R lie on the parallels of colatitude ϑ_i and ϑ_{i+1} respectively. Consider the top boundary $\partial^{\uparrow}\mathsf{R}$. The parallel at ϑ_i on which $\partial^{\uparrow}\mathsf{R}$ lies is a small circle of spherical radius ϑ_i with centre the North Pole.

Now consider the codepoint $\mathbf{a}$. The meridian $\mathbb{O}(\mathbf{a})$ meets the top boundary $\partial^{\uparrow}\mathsf{R}$ at the point $\mathbf{a}_{\uparrow}$, with $\mathsf{s}(\mathbf{a}, \mathbf{a}_{\uparrow}) = \frac{\delta_i}{2}$. The small sphere $\partial\mathsf{S}(\mathbf{a}, \frac{\delta_i}{2})$ meets the parallel at colatitude ϑ_i at exactly one point, which is $\mathbf{a}_\uparrow$. To see this, use stereographic projection to $\mathbb{R}^d$. A similar argument applies to the bottom boundary $\partial^\downarrow \mathsf{R}$. $\square$

Proof of Lemma 4.3.6.

Let $\mathbf{a} := \boxdot\, \mathsf{R}$. By construction we have $\mathbf{a} \notin \partial \mathsf{R}$, so $\mathbf{a} \notin \overline{\partial_\leftrightarrow \mathsf{R}}$, the closure of the side boundary of R.

Let Q be the set of points of $\overline{\partial_\leftrightarrow \mathsf{R}}$ closest to $\mathbf{a}$. More precisely,

$$Q := \{\mathbf{q} \in \overline{\partial_\leftrightarrow \mathsf{R}} \mid \mathsf{s}(\mathbf{a}, \mathbf{q}) \leqslant \mathsf{s}(\mathbf{a}, \mathbf{p}) \text{ for all } \mathbf{p} \in \overline{\partial_\leftrightarrow \mathsf{R}}\}. \tag{4.6.3}$$

The set Q is not empty because $\overline{\partial_\leftrightarrow \mathsf{R}}$ is closed.

Let $\Phi := \mathsf{s}(\mathbf{a}, \partial_\leftrightarrow \mathsf{R})$ and define the spherical cap $C := S(\mathbf{a}, \Phi) \in \mathbb{S}^d$. We must have $\mathsf{s}(\mathbf{a}, \mathbf{q}) = \Phi$ for every point $\mathbf{q} \in Q$, so that $C \cap Q = Q$.

Let $\mathbf{b} \in Q$ and consider the meridian $\varobar\, \mathbf{b}$ and the great circle $B := \overline{\varobar\, \mathbf{b}} \cup - \varobar\, \mathbf{b}$.

Express $\mathbf{a}$ and $\mathbf{b}$ in spherical polar coordinates as

$$\mathbf{a} = \Upsilon(\alpha_1, \ldots, \alpha_d),$$
$$\mathbf{b} = \Upsilon(\beta_1, \ldots, \beta_d),$$

respectively. The constructions of $\mathrm{EQ}(d, \mathcal{N})$ and $\mathrm{EQP}(d, \mathcal{N})$ and the fact that $\mathsf{m}_i \geqslant 2$ imply that $\mathbf{a}$ and $\mathbf{b}$ differ in longitude by at most $\frac{\pi}{2} \pmod{2\pi}$. More precisely, we must have

$$|\alpha_1 - \beta_1| \in \left(0, \frac{\pi}{2}\right] \pmod{2\pi}.$$

Now recall that the longitude of any point of $\varobar\, \mathbf{b}$ is β_1 and the longitude of any point of $- \varobar\, \mathbf{b}$ is $\pi + \beta_1 (\mod 2\pi)$. Therefore $\mathbf{a} \notin \varobar\, \mathbf{b}$ and $\mathbf{a} \notin - \varobar\, \mathbf{b}$. Since $\mathbf{a}$ is not one of the poles, we have $\mathbf{a} \notin B$.

Since $\mathbf{a} \notin B$, the point $\mathbf{a}$ and the great circle B define a great 2-sphere corresponding to $G(\mathbf{b}, \mathbf{a})$ of Lemma 2.3.7. The great 2-sphere $G(\mathbf{b}, \mathbf{a})$ is split by B into two spherical caps each of spherical radius $\frac{\pi}{2}$. Let $\mathbf{e}$ be the centre of one of the two caps. Then the other centre is $-\mathbf{e}$.

We now have two cases.

1. The point $\mathbf{a}$ is one of $\mathbf{e}$ or $-\mathbf{e}$. In this case $\mathsf{s}(\mathbf{a},\mathbf{c}) = \frac{\pi}{2}$ for every point $c \in B$. Since $\mathsf{s}(\mathbf{a}, \partial^{\updownarrow}\mathsf{R}) < \frac{\pi}{2}$, we must have

$$\mathsf{s}(\mathbf{a}, \partial\mathsf{R}) = \mathsf{s}(\mathbf{a}, \partial^{\updownarrow}\mathsf{R}) < \Phi. \tag{4.6.4}$$

2. There is a unique great circle D which contains $\mathbf{a}$, $\mathbf{e}$ and $-\mathbf{e}$. The great circles B and D intersect at right angles at two antipodal points, which we call $\mathbf{d}$ and $-\mathbf{d}$. We have $\mathsf{s}(\mathbf{a},\mathbf{d}) + \mathsf{s}(\mathbf{a},-\mathbf{d}) = \pi$. We cannot have $\mathsf{s}(\mathbf{a},\mathbf{d}) = \mathsf{s}(\mathbf{a},-\mathbf{d})$ because we have excluded case 1. Therefore let $\mathbf{d}$ be the closer of these two points to $\mathbf{a}$. By Lemma 2.3.6 $\mathbf{d}$ is the unique point of B which is closest to $\mathbf{a}$.

   Express the point $\mathbf{d}$ using spherical polar coordinates as $\mathbf{d} = \Upsilon(\delta_1, \ldots, \delta_d)$. We now have two subcases.

   (a) The point $\mathbf{d}$ lies on the open arc $-\oplus\mathbf{b}$. This implies that $\mathbf{d} \neq \mathbf{b}$ and $\mathbf{d} \notin \overline{\partial_{\leftrightarrow}\mathsf{R}}$. We have $\delta_1 = \pi + \beta_1$ and so

   $$|\alpha_1 - \delta_1| > \frac{\pi}{2} \pmod{2\pi}.$$

   Therefore $\mathbf{d}$ lies outside of R. The shortest geodesic arc from $\mathbf{a}$ to $\mathbf{d}$ must pass through $\partial^{\updownarrow}\mathsf{R}$ since if it passed through some point $\mathbf{q} \in \partial_{\leftrightarrow}\mathsf{R}$ we would have $\mathsf{s}(\mathbf{a}, \partial_{\leftrightarrow}\mathsf{R}) < \mathsf{s}(\mathbf{a},\mathbf{d}) < \mathsf{s}(\mathbf{a},\mathbf{b})$, which contradicts the definition of the point $\mathbf{b}$. Therefore (4.6.4) must hold.

   (b) The point $\mathbf{d}$ lies on the closed arc $\overline{\oplus\,\mathbf{b}}$. We have $\oplus\,\mathbf{b} \cap \mathsf{R} \subset \overline{\partial_{\leftrightarrow}\mathsf{R}}$. This again splits into two subcases:

      i. The point $\mathbf{d}$ lies outside R. By the same argument as case 2a we see that (4.6.4) must hold.

      ii. We have $\mathbf{d} = \mathbf{b}$. In this case $\mathsf{s}(\mathbf{a},\mathbf{b}) = \Phi = \mathsf{s}(\mathbf{a}, B)$ and the point $\mathbf{b}$ is the unique point of $B \supset \overline{\oplus\,\mathbf{b}}$ which is closest to $\mathbf{a}$.

By the same arguments as above, for all points $\mathbf{q} \in Q$, either (4.6.4) holds or $\Phi \leqslant \mathsf{s}(\mathbf{a}, \partial^{\updownarrow}\mathsf{R})$ and $C \cap \oplus\, \mathbf{q} = \mathbf{q}$. Therefore either (4.6.4) holds or

$$\mathsf{s}(\mathbf{a}, \partial\mathsf{R}) = \Phi \leqslant \mathsf{s}(\mathbf{a}, \partial^{\updownarrow}\mathsf{R}) \tag{4.6.5}$$

and

$$C \cap \oplus\, \partial_{\leftrightarrow}\mathsf{R} = C \cap \oplus\, Q = Q. \tag{4.6.6}$$

From this point of the proof onward, we exclude the cases where (4.6.4) holds.

The cap C intersects B only at the point $\mathbf{b}$. Since B contains the poles and $\mathbf{b}$ is not a pole, C does not contain either pole.

Now consider the equatorial image of the cap C. By Lemma 2.3.10 we have

$$\Pi\, C = \mathsf{S}^{d-1}(\Pi\, \mathbf{a}, \phi),$$

where

$$\sin\phi := \frac{\sin\Phi}{\sin\alpha_1}. \tag{4.6.7}$$

We also have

$$\Pi\, Q = \Pi(C \cap \oplus\, \partial_{\leftrightarrow}\mathsf{R}) = \Pi\, C \cap \Pi\, \partial_{\leftrightarrow}\mathsf{R} = \Pi\, C \cap \partial\, \Pi\, \mathsf{R},$$

where the last equation is a consequence of Lemma 2.3.13.

Since Q is not empty, $\Pi\, Q$ is not empty, and so $\mathsf{s}(\Pi\, \mathbf{a}, \partial\, \Pi\, \mathsf{R})$ is at most ϕ. But any spherical cap $C' = S(\mathbf{a}, \Phi')$ with $\Phi' < \Phi$ does not intersect $\oplus\, \partial_{\leftrightarrow}\mathsf{R}$ at all, and so $\mathsf{s}(\Pi\, \mathbf{a}, \partial\, \Pi\, \mathsf{R}) = \phi$. $\square$

Proof of Lemma 4.3.7.

The result is an immediate consequence of Lemma 3.5.6, since we have

$$\begin{aligned}\Psi(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i}) &= \sin \frac{\mathcal{T}(-\mathsf{a}_{i-1}, \vartheta_{F,i}) + \mathcal{B}(\mathsf{a}_i, \vartheta)}{2} \ \mathcal{M}(-\mathsf{a}_{i-1}, \mathsf{a}_i, \vartheta_{F,i})^{\frac{1}{1-d}} \\ &= \sin \frac{\vartheta_i + \vartheta_{i+1}}{2} \ \mathsf{m}_i^{\frac{1}{1-d}} = \psi_i.\end{aligned}$$

□

Proof of Lemma 4.3.8.

We use Lemmas 3.5.8 and 3.5.9, which yield

$$\begin{aligned}\Psi(\tau, \beta, \pi - \vartheta) &= \sin \frac{\mathcal{T}(\tau, \pi - \vartheta) + \mathcal{B}(\beta, \pi - \vartheta)}{2} \ \mathcal{M}(\tau, \beta, \pi - \vartheta)^{\frac{1}{1-d}} \\ &= \sin \left(\pi - \frac{\mathcal{B}(\tau, \vartheta - \delta_F) + \mathcal{T}(\beta, \vartheta - \delta_F)}{2}\right) \ \mathcal{M}(\beta, \tau, \vartheta - \delta_F)^{\frac{1}{1-d}} \\ &= \sin \frac{\mathcal{T}(\beta, \vartheta - \delta_F) + \mathcal{B}(\tau, \vartheta - \delta_F)}{2} \ \mathcal{M}(\beta, \tau, \vartheta - \delta_F)^{\frac{1}{1-d}} = \Psi(\beta, \tau, \vartheta - \delta_F).\end{aligned}$$

□

Proof of Lemma 4.3.9.

For $\mathcal{N} > x > N_0(1/2)$, where N_0 is defined by (3.5.8), using the estimate (3.5.49) we have

$$\sin \vartheta_c \geqslant J_c(x) \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} \ \delta_I = K_c'(x) \mathcal{N}^{-\frac{1}{d}},$$

where

$$K_c'(x) := J_c(x) \left(\frac{d \ \omega_d}{\omega_{d-1}}\right)^{\frac{1}{d}}, \tag{4.6.8}$$

with J_c defined by (3.5.45). □

Proof of Lemma 4.3.10.

Throughout this proof, we assume that $\mathcal{N} > x$ where $x \geqslant N_0(5)$, where N_0 is defined by (3.5.8), and x satisfies (3.5.62). We therefore have $\mathsf{n} \geqslant 5$.

We have $(\tau, \beta, \vartheta) \in \mathbb{D}_+ = \mathbb{D}_t \cup \mathbb{D}_{m+}$.

For the top collar, $(\tau, \beta, \vartheta) \in \mathbb{D}_t$, (3.5.38) gives $\tau = 0$, $\beta \in [-1/2, 1/2]$, $\vartheta = \vartheta_c$. Using Lemma 2.3.16 we have

$$\mathcal{V}\left(\mathcal{B}(\beta, \vartheta_c)\right) = \mathcal{V}(\vartheta_c + \delta_F) + \beta \mathcal{V}_R \geqslant \mathcal{V}(\vartheta_c + \delta_F) - \frac{\mathcal{V}_R}{2}.$$

Since $\mathsf{n} \geqslant 5$, we have $\vartheta_c + (1-\eta)\delta_F \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$, and we can use (3.5.67) to obtain

$$\mathcal{V}\left(\mathcal{B}(\beta, \vartheta_c)\right) \geqslant \mathcal{V}(\vartheta_c + \delta_F) - \frac{\mathcal{V}_R}{2} > \mathcal{V}\big(\vartheta_c + (1-\eta)\delta_F\big).$$

Therefore, using Lemma 2.3.16 again, we have

$$\mathcal{B}(\beta, \vartheta_c) > \vartheta_c + (1-\eta)\ \delta_F. \tag{4.6.9}$$

Therefore (3.5.21) and (3.5.24) yield

$$\Delta(\tau, \beta, \vartheta) = \Delta(0, \beta, \vartheta_c) = \mathcal{B}(\beta, \vartheta_c) - \mathcal{T}(0, \vartheta_c) = \mathcal{B}(\beta, \vartheta_c) - \vartheta_c > (1-\eta)\delta_F. \tag{4.6.10}$$

For $(\tau, \beta, \vartheta) \in \mathbb{D}_{m+}$ (3.5.44) gives $\tau \in [-1/2, 1/2]$, $\beta \in [-1/2, 1/2]$, $\vartheta \in [\vartheta_{F,2}, \pi/2 - \delta_F/2]$. Since $\mathsf{n} \geqslant 5$, we have $\vartheta + (1-\eta)\ \delta_F \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$. Using Lemma 2.3.16, (3.5.22) and (3.5.67) we now have

$$\mathcal{V}\big(\mathcal{B}(\beta, \vartheta)\big) = \mathcal{V}(\vartheta + \delta_F) + \beta\mathcal{V}_R \geqslant \mathcal{V}(\vartheta + \delta_F) - \frac{\mathcal{V}_R}{2} > \mathcal{V}\big(\vartheta + (1-\eta)\delta_F\big).$$

Using Lemma 2.3.16 again, we therefore have

$$\mathcal{B}(\beta, \vartheta) > \vartheta + (1-\eta)\delta_F. \tag{4.6.11}$$

Since $\mathsf{n} \geqslant 5$, we have $\vartheta \in [\vartheta_c, \pi - \vartheta_c - \eta\delta_F]$. Using Lemma 2.3.16, (3.5.21) and (3.5.67) we also have

$$\mathcal{V}\big(\mathcal{T}(\tau,\vartheta)\big) = \mathcal{V}(\vartheta) + \tau\mathcal{V}_R \leqslant \mathcal{V}(\vartheta) + \frac{\mathcal{V}_R}{2} < \mathcal{V}(\vartheta + \eta\delta_F),$$

so that

$$\vartheta + \eta\delta_F > \mathcal{T}(\tau,\vartheta). \tag{4.6.12}$$

Combining (4.6.11) and (4.6.12), and using (3.5.24) we therefore have

$$\Delta(\tau,\beta,\vartheta) = \mathcal{B}(\beta,\vartheta) - \mathcal{T}(\tau,\vartheta) > (1-2\eta)\delta_F. \tag{4.6.13}$$

Combining (4.6.10) and (4.6.13) we have

$$\sin\frac{\Delta(\tau,\beta,\vartheta)}{2} > \sin\frac{(1-2\eta)\delta_F}{2}.$$

Define

$$J(x) := \operatorname{sinc}\left(\frac{\rho_H(x)}{2}\left(\frac{\omega_d}{x}\right)^{\frac{1}{d}}\right),$$

with $\rho_H(x)$ defined by (3.5.11).

The estimate (3.5.52) now yields

$$\Delta(\tau,\beta,\vartheta) > K'_\Delta(x)\mathcal{N}^{-\frac{1}{d}},$$

where

$$K'_\Delta(x) := J(x)(1-2\eta)\ \rho_L(x)\ \omega_d^{\frac{1}{d}}, \tag{4.6.14}$$

with $\rho_L(x)$ defined by (3.5.11).

We also have

$$K'_{\Delta}(x) \nearrow K'_{\Delta}(\infty) := (1 - 2\eta)\ \omega_d^{\frac{1}{d}} \tag{4.6.15}$$

as $x \to \infty$, since $\rho_L(x) \nearrow 1$ as $x \to \infty$, by (3.5.12). □

Proof of Lemma 4.3.11.

Throughout this proof, we assume that $\mathcal{N} > x$ where $x \geqslant N_0(5)$, where N_0 is defined by (3.5.8), and x satisfies (3.5.62). We therefore have $\mathsf{n} \geqslant 5$. These assumptions are at least as strong as those used in the proofs of Lemma 3.5.25 and Lemma 4.3.10, so we use parts of those proofs here as well.

We have $(\tau, \beta, \vartheta) \in \mathbb{D}_+ = \mathbb{D}_t \cup \mathbb{D}_{m+}$.

For the top collar, $(\tau, \beta, \vartheta) \in \mathbb{D}_t$, (3.5.38) gives $\tau = 0$, $\beta \in [-1/2, 1/2]$, $\vartheta = \vartheta_c$. Therefore

$$\frac{\mathcal{T}(\tau, \vartheta) + \mathcal{B}(\beta, \vartheta)}{2} = \frac{\vartheta_c + \mathcal{B}(\beta, \vartheta_c)}{2}.$$

From (4.6.11) we have $\mathcal{B}(\beta, \vartheta_c) > \vartheta_c + (1 - \eta)\delta_F$. Therefore

$$\frac{\mathcal{T}(\tau, \vartheta) + \mathcal{B}(\beta, \vartheta)}{2} > \vartheta_c + \frac{1 - \eta}{2}\delta_F > \vartheta_c.$$

Using (3.5.20) we therefore have

$$\begin{aligned}
\Psi(\tau, \beta, \vartheta) = \Psi(0, \beta, \vartheta_c) &= \sin \frac{\vartheta_c + \mathcal{B}(\beta, \vartheta_c)}{2}\ \ \mathcal{M}(0, \beta, \vartheta_c)^{\frac{1}{1-d}} \\
&> \sin \vartheta_c\ \ \mathcal{M}(0, \beta, \vartheta_c)^{\frac{1}{1-d}} = \sin \vartheta_c\ \left(\mathcal{Y}(\vartheta_c) + \beta\right)^{\frac{1}{1-d}} \\
&\geqslant \sin \vartheta_c\ \left(\mathcal{Y}(\vartheta_c) + \frac{1}{2}\right)^{\frac{1}{1-d}}.
\end{aligned}$$

This implies that

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \frac{\sin^{d-1} \vartheta_c}{\mathcal{Y}(\vartheta_c) + \frac{1}{2}}. \tag{4.6.16}$$

Using (2.3.41), (3.2.1), (3.2.3), (3.5.2), (3.5.20) and (3.11.23) and we have

$$\mathcal{Y}(\vartheta_c) = \frac{\mathcal{V}(\vartheta_c + \delta_F) - \mathcal{V}(\vartheta_c)}{\mathcal{V}_R} \leqslant \frac{\delta_F}{\mathcal{V}_R}\ D\mathcal{V}(\vartheta_c + \delta_F) = \frac{\rho}{\delta_I^{d-1}}\ \sin^{d-1}(\vartheta_c + \delta_F). \tag{4.6.17}$$

We therefore define

$$\lambda := \frac{\delta_I^{d-1}}{\rho}, \tag{4.6.18}$$

so that (4.6.16) and (4.6.17) yield

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \lambda \frac{2 \sin^{d-1} \vartheta_c}{2 \sin^{d-1}(\vartheta_c + \delta_F) + \lambda}. \tag{4.6.19}$$

From (3.2.3) and (3.5.10) we have the estimate

$$\lambda \in \left[\rho_H(x)^{-1}, \rho_L(x)^{-1}\right]\ \delta_I^{d-1} = \left[\rho_H(x)^{-1}, \rho_L(x)^{-1}\right]\ \omega_d^{\frac{d-1}{d}}\ \mathcal{N}^{\frac{1-d}{d}}, \tag{4.6.20}$$

where ρ_L and ρ_H are defined by (3.5.11).

We now use (4.6.19) and the estimates (3.5.49), (3.5.57) and (4.6.20) to obtain

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \rho_H(x)^{-1} \omega_d^{\frac{d-1}{d}} \frac{2\ J_c(x)^{d-1} \left(\frac{d}{\omega_{d-1}}\right)^{\frac{d-1}{d}}}{2 \left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + \rho_H(x)\right)^{d-1} + \rho_L(x)^{-1}}\ \mathcal{N}^{\frac{1-d}{d}}.$$

For (τ, β, ϑ) in $\mathbb{D}_t$ we therefore have

$$\Psi(\tau, \beta, \vartheta) \geqslant C'_{\Psi,t}(x)\ \mathcal{N}^{-\frac{1}{d}}, \tag{4.6.21}$$

where

$$C'_{\Psi,t}(x) := \rho_H(x)^{\frac{1}{1-d}} \omega_d^{\frac{1}{d}} \frac{2^{\frac{1}{d-1}}\ J_c(x) \left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}}}{\left(2 \left(\left(\frac{d}{\omega_{d-1}}\right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + \rho_H(x)\right)^{d-1} + \rho_L(x)^{-1}\right)^{\frac{1}{d-1}}}.$$

For $(\tau, \beta, \vartheta) \in \mathbb{D}_{m+}$ (3.5.44) gives $\tau, \beta \in [-1/2, 1/2]$, $\vartheta \in \left[\vartheta_{F,2}, \frac{\pi}{2} - \frac{\delta_F}{2}\right]$.

From (4.6.11) we have $\mathcal{B}(\beta, \vartheta) > \vartheta + (1 - \eta)\delta_F$. From (3.11.6) we have $\mathcal{T}(\tau, \vartheta) > \vartheta - \eta\delta_F$. Therefore

$$\frac{\mathcal{T}(\tau, \vartheta) + \mathcal{B}(\beta, \vartheta)}{2} > \vartheta + \frac{1 - 2\eta}{2}\delta_F > \vartheta.$$

Using (3.5.20) we therefore have

$$\begin{aligned}
\Psi(\tau, \beta, \vartheta) &= \sin \frac{\mathcal{T}(\tau, \vartheta) + \mathcal{B}(\beta, \vartheta)}{2} \quad \mathcal{M}(\tau, \beta, \vartheta)^{\frac{1}{1-d}} \\
&> \sin \vartheta \quad \mathcal{M}(\tau, \beta, \vartheta)^{\frac{1}{1-d}} = \sin \vartheta \, \big(\mathcal{Y}(\vartheta) + \tau + \beta\big)^{\frac{1}{1-d}} \\
&\geqslant \sin \vartheta \, \left(\mathcal{Y}(\vartheta) + 1\right)^{\frac{1}{1-d}}.
\end{aligned}$$

This implies that

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \frac{\sin^{d-1} \vartheta}{\mathcal{Y}(\vartheta) + 1}. \tag{4.6.22}$$

Using (2.3.41), (3.2.1), (3.2.3), (3.5.2), (3.5.20) and (3.11.23) and we have

$$\mathcal{Y}(\vartheta) = \frac{\mathcal{V}(\vartheta + \delta_F) - \mathcal{V}(\vartheta)}{\mathcal{V}_R}, \leqslant \frac{\delta_F}{\mathcal{V}_R} \; D\mathcal{V}(\vartheta + \delta_F) = \frac{\rho}{\delta_I^{d-1}} \; \sin^{d-1}(\vartheta + \delta_F). \tag{4.6.23}$$

From (4.6.18) and (4.6.22) we therefore have

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \frac{\lambda \sin^{d-1} \vartheta}{\sin^{d-1}(\vartheta + \delta_F) + \lambda} =: f(\vartheta). \tag{4.6.24}$$

We now find a lower bound for $f(\vartheta)$ by first showing that $f(\vartheta)$ is monotonic increasing in ϑ, then estimating $f(\vartheta)$ at the low end of the domain.

We have

$$\begin{aligned}
\frac{\partial f(\vartheta)}{\partial \vartheta} &= \frac{\partial}{\partial \vartheta} \frac{\lambda \sin^{d-1} \vartheta}{\sin^{d-1}(\vartheta + \delta_F) + \lambda} \\
&= \frac{\frac{\partial}{\partial \vartheta} \lambda \sin^{d-1} \vartheta}{\sin^{d-1}(\vartheta + \delta_F) + \lambda} + \lambda \sin^{d-1} \vartheta \ \frac{\partial}{\partial \vartheta} \big(\sin^{d-1}(\vartheta + \delta_F) + \lambda \big)^{-1} \\
&= \frac{(d-1)\ \lambda \sin^{d-2} \vartheta \ \cos \vartheta}{\sin^{d-1}(\vartheta + \delta_F) + \lambda} - \sin^{d-1} \vartheta \ \frac{(d-1)\ \lambda \sin^{d-2}(\vartheta + \delta_F)\ \cos \vartheta}{\big(\sin^{d-1}(\vartheta + \delta_F) + \lambda \big)^2} \\
&= \frac{(d-1)\ \lambda \sin^{d-2} \vartheta \ \cos \vartheta}{\big(\sin^{d-1}(\vartheta + \delta_F) + \lambda \big)^2} \left(\sin^{d-2}(\vartheta + \delta_F)\ \big(\sin(\vartheta + \delta_F) - \sin \vartheta \big) + \lambda \right) \\
&= 2 \frac{(d-1)\ \lambda \sin^{d-2} \vartheta \ \cos \vartheta}{\big(\sin^{d-1}(\vartheta + \delta_F) + \lambda \big)^2} \left(\sin^{d-2}(\vartheta + \delta_F)\ \sin \frac{\delta_F}{2} \ \cos \left(\vartheta + \frac{\delta_F}{2} \right) + \lambda \right) \\
&> 0 \text{ for } \vartheta \in \left(0, \frac{\pi}{2} - \frac{\delta_F}{2} \right),
\end{aligned}$$

where the last equality results from Lemma 2.2.1. Therefore from (4.6.24) we have

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant f(\vartheta) \geqslant f(\vartheta_{F,2}) = \frac{\lambda \sin^{d-1} \vartheta_{F,2}}{\sin^{d-1}(\vartheta_{F,3}) + \lambda}. \tag{4.6.25}$$

We now use (4.6.25) and the estimates (3.5.57) and (4.6.20) to obtain

$$\Psi(\tau, \beta, \vartheta)^{d-1} \geqslant \rho_H(x)^{-1} \omega_d^{\frac{d-1}{d}} \frac{J_{F,2}(x)^{d-1} \left(\left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}} + \rho_L(x) \right)^{d-1}}{\left(\left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + 2\rho_H(x) \right)^{d-1} + \rho_L(x)^{-1}} \ \mathcal{N}^{\frac{1-d}{d}}.$$

For (τ, β, ϑ) in $\mathbb{D}_{m+}$ we therefore have

$$\Psi(\tau, \beta, \vartheta) \geqslant C'_{\Psi,m+}(x)\ \mathcal{N}^{-\frac{1}{d}},$$

where

$$C'_{\Psi,m+}(x) := \rho_H(x)^{\frac{1}{1-d}} \omega_d^{\frac{1}{d}} \frac{J_{F,2}(x) \left(\left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}} + \rho_L(x) \right)}{\left(\left(\left(\frac{d}{\omega_{d-1}} \right)^{\frac{1}{d}} J_c(x)^{\frac{1-d}{d}} + 2\rho_H(x) \right)^{d-1} + \rho_L(x)^{-1} \right)^{\frac{1}{d-1}}}. \tag{4.6.26}$$

We now set $C'_{\Psi}(x) := \min(C'_{\Psi,t}(x), C'_{\Psi,m+}(x))$. □

Chapter 5

Separation, discrepancy and energy

> *"... The analytical and geometrical difficulties of the problem of the distribution of the corpuscles when they are arranged in shells are much greater then when they are arranged on rings and I have not yet succeeded in getting a general solution. ..."*
>
> – Thomson, [155, p. 255].

In this chapter we examine when and how, for a given Riesz potential, the well separation and weak-star convergence of a sequence of spherical codes implies that the corresponding sequence of energies converges to the energy double integral. This problem was posed by Saff during the author's visit to Vanderbilt University.

5.1 Energy, weak-star convergence and separation

Later in this chapter we will prove the following theorem.

Theorem 5.1.1. *Let $\mathcal{X}$ be a sequence of $\mathbb{S}^d$ codes which is well separated as per Definition 2.7.2 and weak-star convergent as per Definition 2.11.3. Then for $s \in (0, d)$, the normalized Riesz s energy of the codes in the sequence converges to the normalized spherical double integral of the energy, that is*

$$\mathrm{E}_\ell(\mathcal{X})\mathsf{U}_s \to \mathcal{I}\,\mathsf{U}_s \quad as \quad \ell \to \infty,$$

where E_ℓ and $\mathcal{I}$ are defined by (2.11.6) and (2.11.5) respectively.

Examples.

The following types of sequences of spherical codes satisfy the criteria of Theorem 5.1.1 and are therefore sequences where for $s \in (0, d)$ the s energy converges to the energy double integral.

1. Minimum energy sequences.

   For $s' \in (d-1, d)$ let $\Omega_{s'} = (\Omega_{s',1}, \Omega_{s',2}, \ldots)$ be the sequence of $\mathbb{S}^d$ codes such that $|\Omega_{s',\mathcal{N}}| = \mathcal{N}$ and such that $|\Omega_{s',\mathcal{N}}|$ has the minimum s' energy for $\mathbb{S}^d$ any code with $\mathcal{N}$ codepoints.

   It is known that $\Omega_{s'}$ is both weak-star convergent [93, Chapter 2, 12, pp. 160–162] [39, Theorem 3, p. 236] [69, Theorem 1.1 p. 176] and well separated [89, Theorem 8, p. 179]. Therefore, for $s \in (0, d)$, Theorem 5.1.1 implies that

$$\mathrm{E}_\ell(\Omega_{s'})\mathsf{U}_s \to \mathcal{I}\,\mathsf{U}_s.$$

2. Well-separated sequences of *spherical designs* [73].

   See Section 5.3.

3. Well-separated, diameter-bounded *equal area sequences* [3, 147, 120, 167, 88].

   See Section 5.4.

5.2 Energy, spherical cap discrepancy and separation

The rate of convergence to zero of the normalized spherical cap discrepancy of a well separated sequence of spherical codes imposes a bound on the rate of convergence of the Riesz s energy. This rate is given by the following theorem.

Theorem 5.2.1. *Let $\mathcal{X} = (X_\ell, \ell \in \mathbb{N})$ be a sequence of $\mathbb{S}^d$ codes which is well separated as per Definition 2.7.2 with normalized spherical cap discrepancy converging as $\operatorname{disc}(X_\ell) = \mathrm{O}(\ell^{-a})$, where $\operatorname{disc}(X_\ell)$ is defined by Definition 2.11.5 and $a > 0$. Then for $s \in (0, d)$, there is an upper bound for the Riesz s potential which converges to the energy double integral $\mathcal{I}\,\mathsf{U}_s$ at the rate of at least $\mathrm{O}\left(\ell^{(\frac{s}{d}-1)a}\right)$.*

In other words, if there exist $\mathsf{C}_\Delta, \mathsf{C}_2$ such that each $X_\ell = \{\mathbf{x}_{\ell,1}, \ldots, \mathbf{x}_{\ell,\mathcal{N}_\ell}\} \in \mathcal{X}$ satisfies

$$\|\mathbf{x}_{\ell,i} - \mathbf{x}_{\ell,j}\| > \Delta_\ell := \mathsf{C}_\Delta \; \mathcal{N}_\ell^{-\frac{1}{d}}, \tag{5.2.1}$$

$$\operatorname{disc}(X_\ell) \leqslant \mathsf{C}_2 \; \ell^{-a} \tag{5.2.2}$$

then for the potential U_s we have

$$\mathrm{E}_\ell(\mathcal{X})\mathsf{U}_s \leqslant \mathcal{I}\,\mathsf{U}_s + \mathrm{O}\left(\ell^{\left(\frac{s}{d}-1\right)a}\right). \tag{5.2.3}$$

5.3 Coulomb energy of spherical designs on $\mathbb{S}^2$

Recall from Section 2.9.2 that a spherical t-design is an equal weighted quadrature rule on the unit sphere which is exact for all polynomials of degree up to t. In [73] it was proved that for a well separated sequence of spherical designs on $\mathbb{S}^2$ such that each t-design has $(t+1)^2$ points, the Coulomb energy has the same first term and a second term of the same order as the minimum Coulomb energy for $\mathbb{S}^2$ codes.

This problem was posed by Sloan. The proof in [73] was the joint work of Hesse and the author.

Here we compare this result with the result obtained by combining Theorem 5.2.1 with the bounds on the spherical cap discrepancy of spherical designs obtained by Grabner and Tichy. [64].

First we restate the main results from [73] with notation adjusted to match this thesis.

Theorem 5.3.1. *Let $\mathcal{X}$ be a sequence of spherical designs in $\mathbb{S}^2$ which is well separated with spherical separation constant C_Δ as per Definition 2.7.2. Then the normalized Coulomb energy $\mathrm{E}_t(\mathcal{X})\mathsf{U}_1$ of each spherical design $X_t \in \mathcal{X}$ of cardinality $\mathcal{N}$ and strength t is bounded above by*

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_1 \leqslant 1 + \mathsf{C}_{(\mathsf{C}_\Delta)} \; (t+1)^{-\frac{3}{2}} \; \mathcal{N}^{\frac{1}{4}} - \frac{1}{2}\,\frac{1}{t+\frac{3}{2}} - \frac{1}{2}\,\frac{(t+1)(t+2)}{t+\frac{3}{2}} \; \mathcal{N}^{-1}. \tag{5.3.1}$$

The constant $\mathsf{C}_{(\mathsf{C}_\Delta)} \geqslant 0$ depends on the separation constant C_Δ, but is independent of $\mathcal{N}$ and t.

Theorem 5.3.2. *Let $\mathcal{X}$ be a sequence of spherical designs on $\mathbb{S}^2$, such that for some positive constants μ and C_Δ, if $X_t \in \mathcal{X}$ has cardinality $\mathcal{N} \geqslant 2$ and strength t, then $\mathcal{N} \leqslant \mu(t+1)^2$ and the minimum spherical distance between points of X is bounded below by $\frac{\mathsf{C}_\Delta}{\sqrt{\mathcal{N}}}$. Then the normalized Coulomb energy of each $X_t \in \mathcal{X}$ is bounded above by*

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_1 \leqslant 1 + \mathsf{C}_{(\mathsf{C}_\Delta,\mu)}\ \mathcal{N}^{-\frac{1}{2}}, \tag{5.3.2}$$

where $\mathsf{C}_{(\mathsf{C}_\Delta,\mu)} \geqslant 0$ is independent of $\mathcal{N}$.

Here we recall the well known result that $\mathcal{I}\,\mathsf{U}_1 = 1$. This is also an immediate consequence of Corollaries 2.11.9 and 2.11.10.

As we have mentioned in Section 2.10, it is not yet known whether an infinite sequence of spherical designs exists which satisfies the premise of Theorem 5.3.2. If such a sequence $\mathcal{X}$ exists, Theorem 5.3.2 implies that its Coulomb energy converges to the corresponding energy integral at the rate of $\mathrm{O}(t^{-1})$, that is

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_1 \leqslant 1 + \mathrm{O}(t^{-1}).$$

From [63, Theorem 1] [64, (2.1)] we know that there is a constant C_G such that for any spherical t-design X_t on $\mathbb{S}^2$, we have

$$\operatorname{disc} X_t \leqslant \frac{\mathsf{C}_G}{t+1}.$$

Applying Theorem 5.2.1 to a well separated sequence $\mathcal{X}$ of spherical designs on $\mathbb{S}^2$ such that X_t has strength t, we therefore obtain

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_1 \leqslant 1 + \mathrm{O}(t^{-\frac{1}{2}}).$$

This is a slower rate of convergence than predicted by Theorem 5.3.2, but the result does not depend on the premise of Theorem 5.3.2.

If we use Theorem 5.3.1 with the sequence of spherical designs on $\mathbb{S}^2$ with the lowest known cardinality, that of [86, Theorem 2.3], which has cardinality of $\mathrm{O}(t^3)$, we obtain

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_1 \leqslant 1 + \mathrm{O}(t^{-\frac{3}{4}}).$$

This assumes that the sequence of [86, Theorem 2.3] is well separated. From the construction given in [86, Section 5], this assumption seems reasonable. Thus Theorem 5.3.1 gives a faster rate of convergence for this sequence than is predicted by Theorem 5.2.1.

If instead of the Coulomb energy, we use the Riesz s energy for $s \in (0, 2)$, then for a well separated sequence $\mathcal{X}$ of spherical designs on $\mathbb{S}^2$ such that X_t has strength t, Theorem 5.2.1 yields

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_s \leqslant \mathcal{I}\,\mathsf{U}_s + \mathrm{O}\left(t^{\frac{s}{2}-1}\right). \tag{5.3.3}$$

Hesse [72] has recently generalized the results of [73] to cover this case. The result [72, Theorem 2] implies that for a sequence $\mathcal{X}$ of spherical designs which satisfies the premise of Theorem 5.3.2, the Riesz s for $s \in (0, 2)$ is bounded as

$$\mathrm{E}_t(\mathcal{X})\mathsf{U}_s \leqslant \mathcal{I}\,\mathsf{U}_s + \mathrm{O}\left(t^{s-2}\right). \tag{5.3.4}$$

Again, this result is better than the corresponding result (5.3.3).

5.4 Riesz energy of the EQ codes

Given Theorem 5.1.1, if we can show that $\mathrm{EQP}(d)$, the sequence of EQ codes on $\mathbb{S}^d$, is weak-star convergent, then we can show that the normalized Riesz energy converges to the corresponding energy double integral, since from Theorem 4.3.2 we know that $\mathrm{EQP}(d)$ is well separated. If we can bound the rate of convergence to zero of the spherical cap discrepancy of the codes of $\mathrm{EQP}(d)$, then Theorem 5.2.1 gives us a rate of convergence for the Riesz energy.

5.4.1 Weak-star convergence and spherical cap discrepancy of the EQ codes

The discrepancy theory of Beck and Chen, especially the proofs of [6, Theorem 24A, p. 181], [6, Theorem 24D, p. 182] given at [6, pp. 237–239], hint that we might expect $\mathrm{EQP}(d)$ to be weak-star convergent, with the normalized spherical cap discrepancy of $\mathrm{EQP}(d,\mathcal{N})$ possibly bounded above by $\mathsf{C}\mathcal{N}^{-\frac{1}{2}-\frac{1}{d}}(\log\mathcal{N})^{\frac{1}{2}}$.

Unfortunately, Beck and Chen's Theorem 24D is a non-constructive existence result, and its proof is probabilistic. This is certainly not sufficient to prove that $\mathrm{EQP}(d)$ is weak-star convergent, let alone give a rate of convergence of normalized spherical cap discrepancy.

In this section we give a direct proof that $\mathrm{EQP}(d)$ is weak-star convergent, by estimating the rate of convergence to zero of the normalized spherical cap discrepancy.

It is also possible to prove that $\mathrm{EQP}(d)$ is weak-star convergent by showing that for each polynomial $p \in \mathbb{P}(\mathbb{S}^d)$ that the sequence of quadratures

$$\int_{\mathbb{S}^d} p(\mathbf{x}) d\mathring{\sigma}_\ell(\mathbf{x})$$

converges to the integral

$$\int_{\mathbb{S}^d} p(\mathbf{x}) d\mathring{\sigma}(\mathbf{x}).$$

This is because each polynomial is Lipschitz and because the maximum diameter of the regions of each partition $\mathrm{EQP}(d,\mathcal{N})$ is of order $\mathcal{N}^{-\frac{1}{d}}$. Weak-star convergence follows by appealing to [93, Theorem 0.4, p. 7] and the Stone-Weierstrass theorem as in the case of the proof of Lemma 2.11.6.

Rather than proving weak-star convergence this way, we show that the spherical cap discrepancy of the sequence $\mathrm{EQP}(d)$ converges to zero, and we may then use Lemma 2.11.6 to prove weak-star convergence.

Theorem 5.4.1. *The spherical code* $\mathrm{EQP}(d, \mathcal{N})$ *has spherical cap discrepancy*

$$\mathrm{disc}(d, \mathcal{N}) \leqslant \mathrm{O}(\mathcal{N}^{-\frac{1}{d}}).$$

More precisely,

1. *For* $d = 1$ *we have* $\mathrm{disc}(1, \mathcal{N}) = \mathcal{N}^{-1}$*, and any line in* $\mathbb{R}^2$ *passes through at most two regions of* $\mathrm{EQ}(1, \mathcal{N})$*.*
2. *For* $d > 1$*, there is a constant* C_d *such that any hyperplane in* $\mathbb{R}^{d+1}$ *passes through at most* $\mathsf{C}_d \mathcal{N}^{\frac{d-1}{d}}$ *regions of* $\mathrm{EQ}(d, \mathcal{N})$ *and therefore*

$$\mathrm{disc}(d, \mathcal{N}) \leqslant \mathsf{C}_d \mathcal{N}^{-\frac{1}{d}}. \tag{5.4.1}$$

As a consequence of Theorems 4.3.2, 5.2.1 and 5.4.1, for the potential U_s on the code $\mathrm{EQP}(d, \mathcal{N})$ we have the energy estimate

$$\mathrm{E}_{\mathcal{N}}\left(\mathrm{EQP}(d)\right)\mathsf{U}_s \leqslant \mathcal{I}\,\mathsf{U}_s + \mathrm{O}\left(\mathcal{N}^{\frac{s-d}{d^2}}\right). \tag{5.4.2}$$

5.4.2 Bounds on the s energy of the EQ $\mathbb{S}^2$ codes

For $\mathbb{S}^2$, for $0 < s < 2$, it is possible to show that there is an upper bound for the Riesz s energy of the EQ $\mathbb{S}^2$ codes which converges to the energy integral at a faster rate than that predicted by the combination of Theorems 5.2.1 and 5.4.1, which (5.4.2) gives as $\mathrm{O}\left(\mathcal{N}^{\frac{s-2}{4}}\right)$. This is the subject of the next theorem.

Theorem 5.4.2. *Let* $\mathcal{Z}$ *be a sequence of equal area partitions of* $\mathbb{S}^2$ *as per Definition 2.4.1 such that* $|Z_{\mathcal{N}}| = \mathcal{N}$ *for each partition* $Z_{\mathcal{N}}$ *in the sequence and such that* $\mathcal{Z}$ *is diameter bounded as per Definition 2.4.3. Specifically, for each* $\mathsf{R} \in Z_{\mathcal{N}}$*, we require*

$$\mathrm{diam}(\mathsf{R}) \leqslant \oslash_{\mathcal{N}} := \frac{\mathsf{C}_{\oslash}}{\sqrt{\mathcal{N}}}. \tag{5.4.3}$$

Let $\mathcal{X}$ *be a well separated sequence of* $\mathbb{S}^2$ *codes with separation constant* C_{Δ} *such that for each* $X_{\mathcal{N}}$ *in the sequence,* $|X_{\mathcal{N}}| = \mathcal{N}$ *and such that each codepoint of* $X_{\mathcal{N}}$ *is* *contained in a distinct region of the partition $Z_{\mathcal{N}}$. Specifically, for each* $\mathbf{x}, \mathbf{y} \in X_{\mathcal{N}}$ *such that* $\mathbf{x} \neq \mathbf{y}$, *we require*

$$\|\mathbf{x} - \mathbf{y}\| \geqslant \Delta_{\mathcal{N}} := \frac{\mathsf{C}_{\Delta}}{\sqrt{\mathcal{N}}}. \tag{5.4.4}$$

Then for $s \in (0, 2)$, *the Riesz* s *energy* $\mathrm{E}_{\mathcal{N}}(\mathcal{X})\mathsf{U}_s$ *as per (2.11.6) satisfies the upper bound*

$$\mathrm{E}_{\mathcal{N}}(\mathcal{X})\mathsf{U}_s \leqslant \begin{cases} \frac{2^{1-s}}{2-s} + \frac{2^{1-s}\ \mathsf{C}_{\oslash}}{1-s}\mathcal{N}^{-\frac{1}{2}} + \mathrm{O}\left(\mathcal{N}^{\frac{s}{2}-1}\right) & (s < 1), \\ 1 + \frac{\mathsf{C}_{\oslash}}{8}\mathcal{N}^{-\frac{1}{2}}\log(\mathcal{N}) + \mathrm{O}\left(\mathcal{N}^{-\frac{1}{2}}\right) & (s = 1), \\ \frac{2^{1-s}}{2-s} + \mathrm{O}\left(\mathcal{N}^{\frac{s}{2}-1}\right) & (s > 1). \end{cases}$$

As an immediate consequence of Corollaries 2.11.9 and 2.11.10 we have

$$\mathcal{I}\,\mathsf{U}_s = \frac{2^{1-s}}{2-s}.$$

5.4.3 Numerical results for $d-1$ energy of the EQ codes

The author ran the EQSP Matlab function `eq_point_set` to create the codes $\mathrm{EQP}(d, \mathcal{N})$ for d from 2 to 4 and $\mathcal{N}$ from 2 to 20 000, and then ran the function `point_set_energy_dist` to determine the Riesz $d-1$ energy of these codes.

The results are plotted in this section. The normalized energy is plotted as Figure 5.1, with the energy for $\mathrm{EQP}(2)$ as blue crosses, $\mathrm{EQP}(3)$ as red crosses, $\mathrm{EQP}(4)$ as green crosses.

From (5.4.2) we should expect the normalized energy to converge to the normalized energy integral. From Corollaries 2.11.9 and 2.11.10 we see that this integral evaluates to 1. In Figure 5.1 not only does the normalized energy appear to converge to 1 in all three cases, it is always less than 1, and the difference appears to be systematic.

In Figure 5.2 we examine this difference more closely. Again, we have $\mathrm{EQP}(2)$ as blue crosses, $\mathrm{EQP}(3)$ as red crosses, $\mathrm{EQP}(4)$ as green crosses.

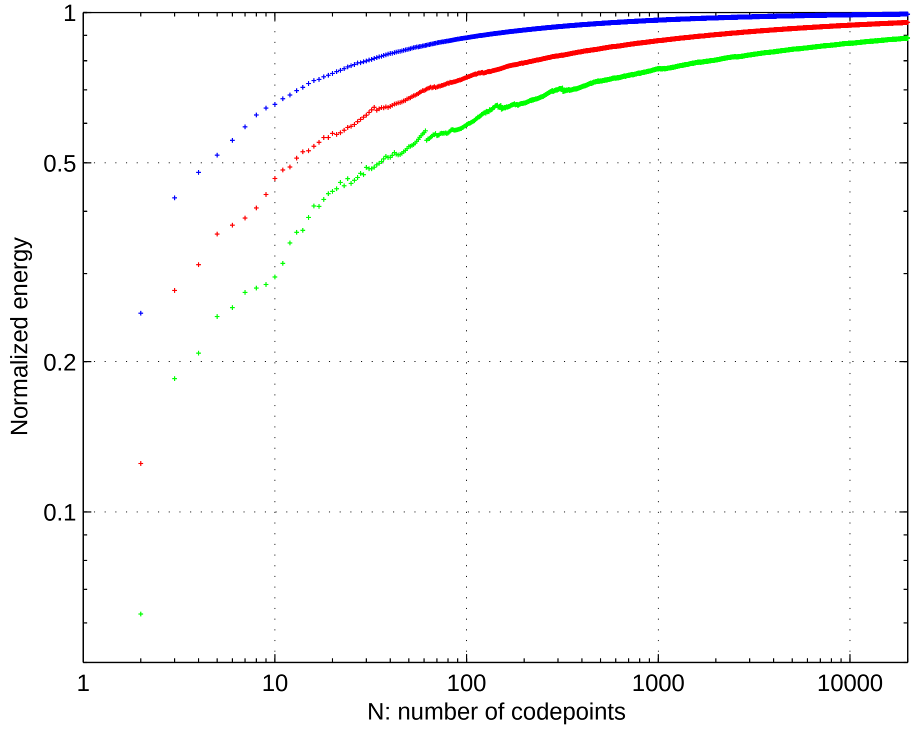

Figure 5.1: Normalized energy of $\mathrm{EQP}(d, \mathcal{N})$ (log-log scale)

By plotting the difference on a log-log scale we have revealed that in each of the three cases the difference is approximately of the form $a\mathcal{N}^b$ for some a, b. We now guess that the asymptotic expansion of the normalized $d-1$ energy for $\mathrm{EQP}(d, \mathcal{N})$ has the same form as the upper bound given in [88, Theorem 1, (1.6), p. 524] for the minimum $d-1$ energy for $\mathbb{S}^d$ codes with $\mathcal{N}$, when suitably normalized. That is, we guess that

$$\mathrm{E}\left(\mathrm{EQP}(d, \mathcal{N})\right)\ \mathsf{U}_{d-1} \simeq 1 - \mathsf{C}_d \mathcal{N}^{-\frac{1}{d}}$$

for some positive constant C_d. To test this assumption, we see that we should examine the value of the energy coefficient $\mathrm{ec_d}$ defined by

$$\mathrm{ec}_d(\mathcal{N}) := \left(1 - \mathrm{E}\left(\mathrm{EQP}(d, \mathcal{N})\right)\ \mathsf{U}_{d-1}\right) \mathcal{N}^{\frac{1}{d}}. \tag{5.4.5}$$

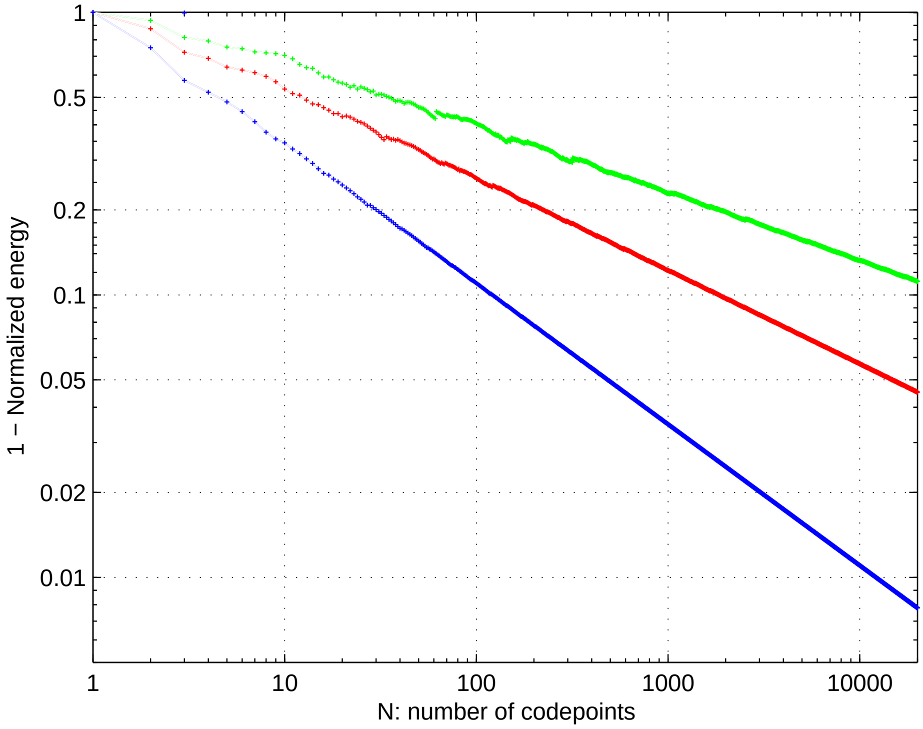

Figure 5.2: 1 minus normalized energy of $\mathrm{EQP}(d, \mathcal{N})$ (log-log scale)

Remarks. Please note that $\mathrm{ec}_d(\mathcal{N})$ is the same as `-2*eq_energy_coeff(`d`,`$\mathcal{N}$`)` as per the EQSP Matlab code version 1.10.

Figures 5.3, 5.4 and 5.5 show $\mathrm{ec}_d(\mathcal{N})$ for $\mathcal{N}$ from 2 to 20 000, for d from 2 to 4 respectively.

On examining $\mathrm{E}\left(\mathrm{EQP}(d, \mathcal{N})\right) \mathsf{U}_{d-1}$ more carefully, we see that for $\mathcal{N} \in \{2, \ldots, 20000\}$ we have

$$\mathrm{E}\left(\mathrm{EQP}(2, \mathcal{N})\right) \ \mathsf{U}_1 = 1 - 1.1025\mathcal{N}^{-\frac{1}{2}} + 0.18\mathcal{N}^{-\frac{3}{2}} \pm 0.082\mathcal{N}^{-1}, \tag{5.4.6}$$

$$\mathrm{E}\left(\mathrm{EQP}(3, \mathcal{N})\right) \ \mathsf{U}_2 = 1 - 1.231\mathcal{N}^{-\frac{1}{3}} + 0.04\mathcal{N}^{-\frac{2}{3}} - 0.38\mathcal{N}^{-1} \pm 0.26\mathcal{N}^{-\frac{5}{6}}, \tag{5.4.7}$$

$$\mathrm{E}\left(\mathrm{EQP}(4, \mathcal{N})\right) \ \mathsf{U}_3 = 1 - 1.355\mathcal{N}^{-\frac{1}{4}} + 0.3\mathcal{N}^{-\frac{1}{2}} \pm 0.83\mathcal{N}^{-\frac{3}{4}}. \tag{5.4.8}$$

Remarks. This is an empirical observation only, based on the output of the EQSP Maple code, version 1.10. It is unknown whether this

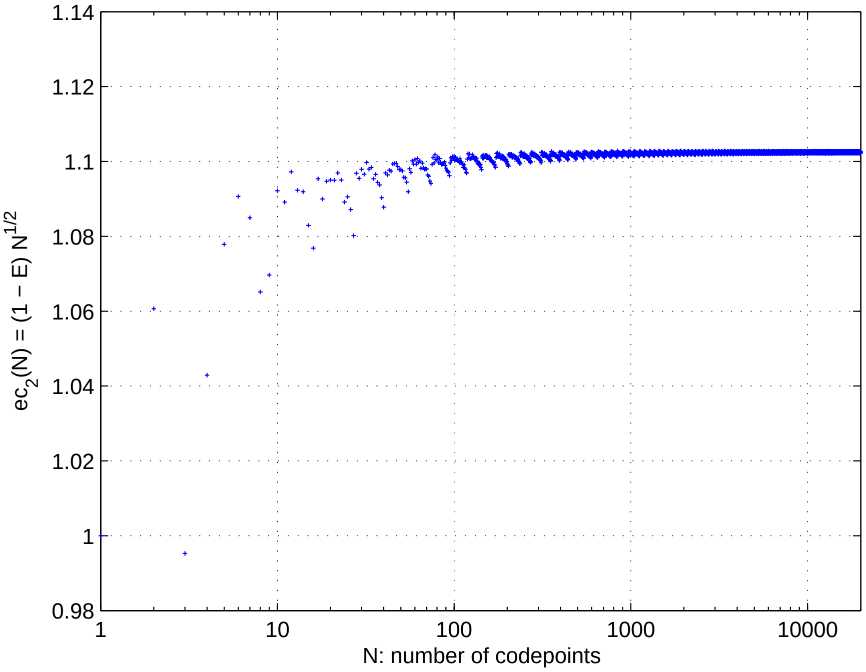

Figure 5.3: Energy coefficient of EQP$(2, \mathcal{N})$ (semi-log scale)

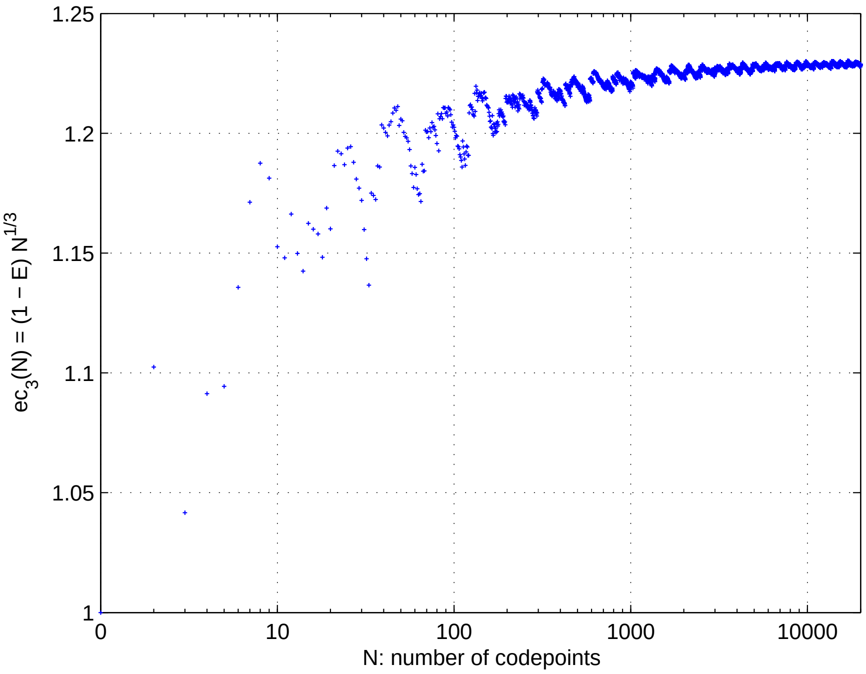

Figure 5.4: Energy coefficient of EQP$(3, \mathcal{N})$ (semi-log scale)

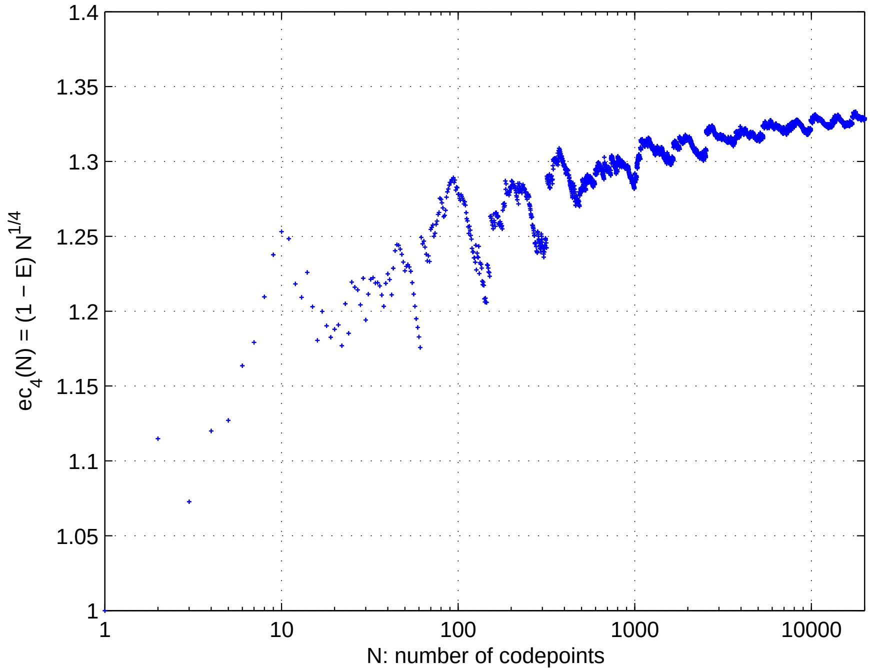

Figure 5.5: Energy coefficient of $\mathrm{EQP}(4,\mathcal{N})$ (semi-log scale)

observation holds for $\mathcal{N} > 20000$. Also, there is no theoretical justification for the higher order terms, including the $\pm$ term.

The best known theoretical upper bound for the minimum energy for the Coulomb potential U_1 on $\mathbb{S}^2$ is currently the one implied by [120, Theorems 2.2 and 2.3, pp. 649–650], which is

$$\mathcal{E}_2(\mathcal{N})\ \mathsf{U}_1 := \min_{X \subset \mathbb{S}^2, |X| = \mathcal{N}} \mathrm{E}(X)\ \mathsf{U}_1 \leqslant 1 - \frac{1}{7}\mathcal{N}^{-\frac{1}{2}}. \tag{5.4.9}$$

The bound given at [134, p. 8] is a misprint. The constant is out by a factor of 2.

A conjecture of Kuijlaars and Saff [88, Conjecture 2, p. 528] implies that $\mathcal{E}_2(\mathcal{N})\mathsf{U}_1$ has a conjectured asymptotic expansion of

$$\mathcal{E}_2(\mathcal{N})\ \mathsf{U}_1 = 1 + 6\left(\frac{\sqrt{3}}{8\pi}\right)^{\frac{1}{2}} \zeta\left(\frac{1}{2}\right) L_{-3}\left(\frac{1}{2}\right)\mathcal{N}^{-\frac{1}{2}} + o(\mathcal{N}^{-\frac{1}{2}}). \tag{5.4.10}$$

The constant for the second term is negative and is approximately 1.106102567...

5.5 Proofs

5.5.1 Energy, weak-star convergence and separation

Proof of Theorem 5.1.1.

Cutoff potential.

Fix $s \in (0, d)$. Since s is fixed, it is convenient to define the abbreviation

$$\mathsf{U}(r) := \mathsf{U}_s(r) = r^{-s}, \tag{5.5.1}$$

Now fix a weak-star convergent and well separated sequence $\mathcal{X}$, with separation constant C_Δ as per Definition 2.7.2. That is, for $\mathbf{x}, \mathbf{y} \in X_\ell$,

$$\|\mathbf{x} - \mathbf{y}\| > \Delta_\ell := \mathsf{C}_\Delta \mathcal{N}_\ell^{-\frac{1}{d}}. \tag{5.5.2}$$

We now define the cutoff potential U_ℓ by

$$U_\ell(r) := \begin{cases} \Delta_\ell^{-s} & (0 < r \leqslant \Delta_\ell) \\ \mathsf{U}(r) = r^{-s} & (r > \Delta_\ell). \end{cases}$$

By the triangle inequality, for any $\ell, m \in \mathbb{N}$,

$$|\mathrm{E}_\ell(\mathcal{X})\mathsf{U} - \mathcal{I}\,\mathsf{U}| \leqslant a_\ell + b_{\ell,m} + c_{\ell,m} + d_{\ell,m} + e_m,$$

where

$$a_\ell := |\mathrm{E}_\ell\,\mathsf{U} - \mathrm{E}_\ell\,U_\ell|\,, \quad b_{\ell,m} := |\mathrm{E}_\ell\,U_\ell - \mathrm{E}_\ell\,U_m|\,, \quad c_{\ell,m} := |\mathrm{E}_\ell\,U_m - \mathcal{I}_\ell\,U_m|\,,$$

$$d_{\ell,m} := |\mathcal{I}_\ell\,U_m - \mathcal{I}\,U_m|\,, \quad e_m := |\mathcal{I}\,U_m - \mathcal{I}\,\mathsf{U}|\,.$$

Here and in the following argument, we have dropped the dependence of E_ℓ and $\mathcal{I}_\ell$ on $\mathcal{X}$, since we have fixed $\mathcal{X}$.

The proof proceeds by showing that each of these terms converges.

Since $\mathcal{X}$ is weak-star convergent, the function $\mathcal{L}$ defined by (2.11.2) is well defined. We will need this function in what follows.

Convergence of a_ℓ*.*

Since $(\mathsf{U} - U_\ell)(r) = 0$ for $r \geqslant \Delta_\ell$, the separation condition guarantees that $a_\ell = 0$ for any ℓ. In detail,

$$\begin{aligned}
a_\ell = |\mathrm{E}_\ell(\mathsf{U} - U_\ell)| &= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} (\mathsf{U} - U_\ell)(\|\mathbf{x} - \mathbf{y}\|) \; d\mathring{\sigma}_\ell[\mathbf{x}](\mathbf{y}) \; d\mathring{\sigma}_\ell(\mathbf{x}) \\
&= \int_{\mathbb{S}^d} \Bigg(\int_{\|\mathbf{x}-\mathbf{y}\| \leqslant \Delta_\ell} (\mathsf{U} - U_\ell)(\|\mathbf{x} - \mathbf{y}\|) \; d\mathring{\sigma}_\ell[\mathbf{x}](\mathbf{y}) \\
&\qquad + \int_{\|\mathbf{x}-\mathbf{y}\| > \Delta_\ell} (\mathsf{U} - U_\ell)(\|\mathbf{x} - \mathbf{y}\|) \; d\mathring{\sigma}_\ell[\mathbf{x}](\mathbf{y}) \Bigg) \; d\mathring{\sigma}_\ell(\mathbf{x}) \\
&= \int_{\mathbb{S}^d} \Bigg(\int_{\|\mathbf{x}-\mathbf{y}\| \leqslant \Delta_\ell} (\mathsf{U}(\|\mathbf{x} - \mathbf{y}\|) - \Delta_\ell) \; d\mathring{\sigma}_\ell[\mathbf{x}](\mathbf{y}) + \int_{\|\mathbf{x}-\mathbf{y}\| > \Delta_\ell} 0 \; d\mathring{\sigma}_\ell[\mathbf{x}](\mathbf{y}) \Bigg) \; d\mathring{\sigma}_\ell(\mathbf{x}) \\
&= \frac{1}{\mathcal{N}_\ell^2} \sum_{k=1}^{\mathcal{N}_\ell} \sum_{\substack{j=1 \\ \mathbf{x}_{\ell,j} \neq \mathbf{x}_{\ell,k} \\ \|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,j}\| \leqslant \Delta_\ell}}^{\mathcal{N}_\ell} \big(\mathsf{U}(\|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,j}\|) - \Delta_\ell \big) = 0.
\end{aligned}$$

Convergence of $b_{\ell,m}$*.*

For $\mathcal{N}_m \geqslant \mathcal{N}_\ell$, we can show that $b_{\ell,m} = 0$ by the same argument as for a_ℓ. Therefore we assume $\mathcal{N}_m < \mathcal{N}_\ell$.

Recall that

$$b_{\ell,m} = |\mathrm{E}_\ell(U_\ell - U_m)| = \left| \frac{1}{\mathcal{N}_\ell} \sum_{k=1}^{\mathcal{N}_\ell} \frac{1}{\mathcal{N}_\ell} \sum_{\substack{j=1 \\ j \neq k}}^{\mathcal{N}_\ell} (U_\ell - U_m)(\|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,j}\|) \right|. \tag{5.5.3}$$

Fix any k in $\{1, \ldots, \mathcal{N}_\ell\}$ and define

$$\mathrm{E}_{\ell,k}\, u := \int_{\mathbb{S}^d} u(\|\mathbf{x}_{\ell,k} - \mathbf{y}\|) \; d\mathring{\sigma}_\ell[\mathbf{x}_{\ell,k}](\mathbf{y}) = \frac{1}{\mathcal{N}_\ell} \sum_{\substack{j=1 \\ j \neq k}}^{\mathcal{N}_\ell} u(\|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,j}\|)$$

for any potential $u : (0, 2] \to \mathbb{R}$.

The sum within (5.5.3) can therefore be expressed as

$$\mathrm{E}_\ell(U_\ell - U_m) = \frac{1}{\mathcal{N}_\ell} \sum_{k=1}^{\mathcal{N}_\ell} \mathrm{E}_{\ell,k}(U_\ell - U_m). \tag{5.5.4}$$

We now use the punctured measure $\mathring{\sigma}_\ell[\mathbf{x}_{\ell,k}]$ to define the normalized counting function $g_{\ell,k}$ by

$$g_{\ell,k}(R) := \mathring{\sigma}_\ell[\mathbf{x}_{\ell,k}]\ S\big(\mathbf{x}_{\ell,k}, \Upsilon^{-1}(R)\big) = \frac{\big|X_\ell \cap \mathsf{S}(\mathbf{x}_{\ell,k}, \Upsilon^{-1}(R))\big| - 1}{\mathcal{N}_\ell}. \tag{5.5.5}$$

We can use $g_{\ell,k}$ to express $\mathrm{E}_{\ell,k}\, u$ as

$$\mathrm{E}_{\ell,k}\, u = \frac{1}{\mathcal{N}_\ell} \sum_{\substack{j=1 \\ j \neq k}}^{\mathcal{N}_\ell} u\left(\|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,j}\|\right) = \int_0^2 u(r)\ dg_{\ell,k}(r) = \int_{\Delta_\ell}^2 u(r)\ dg_{\ell,k}(r), \tag{5.5.6}$$

where the last equation is a result of the separation condition (5.5.2).

We therefore have

$$\begin{aligned} \mathrm{E}_{\ell,k}(U_\ell - U_m) &= \int_{\Delta_\ell}^2 U_\ell(r) - U_m(r)\ dg_{\ell,k}(r) \\ &= \begin{cases} 0 & (\mathcal{N}_m \geqslant \mathcal{N}_\ell) \\ \int_{\Delta_\ell}^{\Delta_m} r^{-s} - \Delta_m^{-s}\ dg_{\ell,k}(r). & (\mathcal{N}_m < \mathcal{N}_\ell) \end{cases} \end{aligned}$$

Set $f(r) := r^{-s} - \Delta_m^{-s}$. Note that $f(r) \geqslant 0$ for $r \in [\Delta_\ell, \Delta_m]$. We therefore have

$$\mathrm{E}_{\ell,k}(U_\ell - U_m) = \int_{\Delta_\ell}^{\Delta_m} f(r)\ dg_{\ell,k}(r). \tag{5.5.7}$$

We recognize (5.5.7) as a Riemann–Stieltjes integral, and since f is a differentiable radial function, we can re-express (5.5.7) as

$$
\begin{aligned}
\mathrm{E}_{\ell,k}(U_\ell - U_m) &= [f(r)g_{\ell,k}(r)]_{\Delta_\ell}^{\Delta_m} - \int_{\Delta_\ell}^{\Delta_m} Df(r)g_{\ell,k}(r)\ dr \\
&= f(\Delta_m)g_{\ell,k}(\Delta_m) - f(\Delta_\ell)g_{\ell,k}(\Delta_\ell) + \int_{\Delta_\ell}^{\Delta_m} -Df(r)g_{\ell,k}(r)\ dr \\
&= \int_{\Delta_\ell}^{\Delta_m} -Df(r)g_{\ell,k}(r)\ dr \quad \text{for } \mathcal{N}_\ell > \mathcal{N}_m,
\end{aligned}
$$

since $f(\Delta_m) = 0$ and since the separation condition (5.5.2) gives $g_{\ell,k}(\Delta_\ell) = 0$.

Define $\Theta_\ell := \Upsilon^{-1}(\Delta_\ell) = 2\sin^{-1}(\Delta_\ell/2)$. The separation condition (5.5.2) implies that we can place each node $\mathbf{x}_{\ell,j}$ of X_ℓ in a spherical cap $S\left(\mathbf{x}_{\ell,j}, \frac{\Theta_\ell}{2}\right)$ with no two caps overlapping. By (2.7.14) we therefore have

$$
g_{\ell,k}(R) \leqslant 4^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} \left(\frac{R}{\Delta_\ell}\right)^d \mathcal{N}_\ell^{-1} = \frac{4}{\mathsf{C}_\Delta}^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} R^d =: h(R). \tag{5.5.8}
$$

Now, for $r \in [\Delta_\ell, \Delta_m]$ we note that $-Df(r) = sr^{-s-1} > 0$, so

$$
0 \leqslant \int_{\Delta_\ell}^{\Delta_m} -Df(r)g_{\ell,k}(r)\ dr \leqslant \int_{\Delta_\ell}^{\Delta_m} -Df(r)h(r)\ dr.
$$

Thus

$$
|\mathrm{E}_{\ell,k}(U_\ell - U_m)| \leqslant \int_{\Delta_\ell}^{\Delta_m} -Df(r)h(r)\ dr = A \int_{\Delta_\ell}^{\Delta_m} r^{d-s-1}\ dr,
$$

where

$$
A := \frac{4}{\mathsf{C}_\Delta}^d \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)}\ s.
$$

We therefore have

$$\begin{aligned}|\mathrm{E}_{\ell,k}(U_\ell - U_m)| &\leqslant A\int_{\Delta_\ell}^{\Delta_m} r^{d-s-1}\ dr = \frac{A}{d-s}(\Delta_m^{d-s} - \Delta_\ell^{d-s}) \\ &\leqslant \frac{A}{d-s}\ \mathsf{C}_\Delta^{d-s}(\mathcal{N}_m^{\frac{s}{d}-1} - \mathcal{N}_\ell^{\frac{s}{d}-1}) \leqslant \frac{A}{d-s}\ \mathsf{C}_\Delta^{d-s}\mathcal{N}_m^{\frac{s}{d}-1}.\end{aligned}$$

Since $\frac{s}{d} - 1 < 0$, if $m \geqslant \mathcal{L}(N_b)$ we therefore have

$$|\mathrm{E}_{\ell,k}(U_\ell - U_m)| \leqslant \frac{A}{d-s}\ \mathsf{C}_\Delta^{d-s} N_b^{\frac{s}{d}-1}.$$

We have therefore shown that $|\mathrm{E}_{\ell,k}(U_\ell - U_m)| < \frac{\epsilon}{4}$ for $m \geqslant \mathcal{L}(N_b)$ where N_b is the smallest positive integer such that $\frac{A}{d-s}\ \mathsf{C}_\Delta^{d-s} N_b^{\frac{s}{d}-1} < \frac{\epsilon}{4}$.

Since this bound is independent of our codepoint index k, we must also have $b_{\ell,m} < \frac{\epsilon}{4}$, for $m \geqslant \mathcal{L}(N_b)$.

Convergence of $c_{\ell,m}$.

The term $c_{\ell,m}$ is just the diagonal of the double sum. That is,

$$\begin{aligned}c_{\ell,m} &= \frac{1}{{\mathcal{N}_\ell}^2}\sum_{k=1}^{\mathcal{N}_\ell} U_m(\|\mathbf{x}_{\ell,k} - \mathbf{x}_{\ell,k}\|) \\ &= \frac{U_m(0)}{\mathcal{N}_\ell} = \frac{\Delta_m^{-s}}{\mathcal{N}_\ell} = \mathsf{C}_\Delta^{-s}{\mathcal{N}_m}^{\frac{s}{d}}{\mathcal{N}_\ell}^{-1} \\ &< \frac{\epsilon}{4} \quad \text{if} \quad \ell \geqslant \mathcal{L}\left(N_c(m)\right),\end{aligned}$$

where

$$N_c(m) := 4\mathsf{C}_\Delta^{-s}{\mathcal{N}_m}^{\frac{s}{d}}\epsilon^{-1} + 1.$$

Convergence of $d_{\ell,m}$.

For any m, the weak-star convergence of $\mathcal{X}$ ensures that

$$d_{\ell,m} \to 0 \quad \text{as } \ell \to \infty,$$

since weak convergence of $\mathring{\sigma}_\ell \to \mathring{\sigma}$ implies weak convergence of $\mathring{\sigma}_\ell \times \mathring{\sigma}_\ell \to \mathring{\sigma} \times \mathring{\sigma}$, because $\mathbb{S}^d$ is separable [9, Theorem 3.2], [11, Theorem 2.8]. So we have $d_{\ell,m} < \frac{\epsilon}{4}$ if $\ell \geqslant \mathcal{L}\left(N_d(m)\right)$, where $N_d(m)$ depends on ϵ, m and X.

Convergence of e_m.

We show below that $e_m < \frac{\epsilon}{4}$ if $m \geqslant \mathcal{L}(N_e)$, where N_e is defined using ϵ, d, s and C_Δ.

$$
\begin{aligned}
e_m = |\mathcal{I}(U_m - \mathsf{U})| &= \left| \int_0^2 (U_m - \mathsf{U})(r)\ r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1} dr \right| \\
&= \int_0^{\Delta_m} \big(\mathsf{U}(r) - U_m(r)\big)\ r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1} dr \\
&= \int_0^{\Delta_m} (r^{-s} - \Delta_m^{-s})\ r^{d-1} \left(1 - \frac{r^2}{4}\right)^{\frac{d}{2}-1} dr \\
&= \mathcal{J}(\Delta_m)\ \mathsf{U} - \Delta_m^{-s} \mathring{\mathcal{V}}_E(\Delta_m) \\
&\leqslant \mathcal{J}(\Delta_m)\ \mathsf{U} \leqslant \mathsf{C}_{d,H} \frac{\Delta_m^{d-s}}{d-s} \leqslant \mathsf{C}_\Delta^{d-s} \frac{\mathsf{C}_\Delta^{d-s}}{d-s} \mathcal{N}_m{}^{\frac{s}{d}-1}.
\end{aligned}
$$

Note that $\frac{s}{d} - 1 < 0$, since $s < d$. Now set

$$
N_e := \left(\frac{4}{\epsilon} \frac{\mathsf{C}_{d,H}}{d-s} \right)^{\frac{d}{d-s}} \mathsf{C}_\Delta^d
$$

Then for $m \geqslant \mathcal{L}(N_e)$, $e_m < \frac{\epsilon}{4}$.

Reassembly.

So we have $|\mathrm{E}_\ell\, \mathsf{U} - \mathcal{I}\, \mathsf{U}| < \epsilon$ if $\ell \geqslant \mathcal{L}\left(\max\left(N_c(M), N_d(M)\right)\right)$, where $M := \mathcal{L}\left(\max(N_b, N_e)\right)$. In other words, $|\mathrm{E}_\ell\, \mathsf{U} - \mathcal{I}\, \mathsf{U}| \to 0$ as $\ell \to \infty$. □

5.5.2 Energy, spherical cap discrepancy and separation

Proof of Theorem 5.2.1.

Fix $s \in (0, d)$ and fix a sequence $\mathcal{X}$ having the required properties.

As in the proof of Theorem 5.1.1, we use the abbreviations $\mathrm{E}_\ell := \mathrm{E}_\ell(\mathcal{X})$ and $\mathsf{U} := \mathsf{U}_s$.

We calculate the energy E_ℓ using a sum of Riemann-Stieltjes integrals, one for each of the $\mathcal{N}_\ell$ nodes. We use the punctured normalized counting function $g_{\ell,k}(r)$

defined by (5.5.5). Then

$$\mathrm{E}_\ell\, \mathsf{U} = \sum_{k=1}^{\mathcal{N}_\ell} \mathrm{E}_{\ell,k}\, \mathsf{U}$$

where for any potential $u : (0,2] \to \mathbb{R}$, we have as per (5.5.6),

$$\mathrm{E}_{\ell,k}\, u := \int_0^2 u(r)\ dg_{\ell,k}(r) = \int_{\Delta_\ell}^2 u(r)\ dg_{\ell,k}(r).$$

If u is differentiable on $(\Delta_\ell, 2]$ we then have

$$\mathrm{E}_{\ell,k}\, u = [u(r) g_{\ell,k}(r)]_{\Delta_\ell}^2 - \int_{\Delta_\ell}^2 Du(r)\ g_{\ell,k}(r)\ dr = u(2)\ (1 - \mathcal{N}_\ell^{-1}) - \int_{\Delta_\ell}^2 Du(r)\ g_{\ell,k}(r)\ dr.$$

Since $\mathsf{U}(r) = r^{-s}$, we have $D\mathsf{U}(r) = -sr^{-s-1}$, and so

$$\mathrm{E}_{\ell,k}\, \mathsf{U} = 2^{-s}(1 - \mathcal{N}_\ell^{-1}) + s\ \int_{\Delta_\ell}^2 r^{-s-1}\ g_{\ell,k}(r)\ dr. \tag{5.5.9}$$

We use the standard packing argument of Lemma 2.7.8 to show that there is some $\mathsf{C}_3 \geqslant 1$ such that

$$g_{\ell,k}(r) \leqslant \mathsf{C}_3\ r^d, \tag{5.5.10}$$

where

$$\mathsf{C}_3 := \frac{\mathsf{C}_{H,d}}{\mathsf{C}_{L,d}(1)} \left(\frac{4}{\mathsf{C}_\Delta}\right)^d.$$

From the normalized spherical cap discrepancy and Lemma 2.3.21 we also know that

$$g_{\ell,k}(r) \leqslant \overset{*}{\mathcal{V}}_E(r) + \operatorname{disc}(X_\ell) \leqslant \overset{*}{\mathcal{V}}_E(r) + \mathsf{C}_2\ \ell^{-a} \leqslant \mathsf{C}_4\ r^d + \mathsf{C}_2\ \ell^{-a}, \tag{5.5.11}$$

where

$$\mathsf{C}_4 := \frac{\mathsf{C}_{H,d}}{d}. \tag{5.5.12}$$

We now find the point r_e where these two upper bounds are equal. We have

$$\mathsf{C}_3\ r_e^d = \mathsf{C}_4\ r_e^d + \mathsf{C}_2\ \ell^{-a},$$

so $(\mathsf{C}_3 - \mathsf{C}_4)\ r_e^d = \mathsf{C}_2\ \ell^{-a}$, and therefore we define

$$r_e := \mathsf{C}_5\ \ell^{-\frac{a}{d}}, \tag{5.5.13}$$

where

$$\mathsf{C}_5 := \left(\frac{\mathsf{C}_2}{\mathsf{C}_3 - \mathsf{C}_4}\right)^{\frac{1}{d}}. \tag{5.5.14}$$

We now have

$$g_{\ell,k}(r) \leqslant h(r) := \begin{cases} 0, & r \in [0, \Delta_\ell] \\ \mathsf{C}_3\ r^d, & r \in (\Delta_\ell, r_e) \\ \overset{*}{\mathcal{V}}_E(r) + \mathsf{C}_2\ \ell^{-a}, & r \in [r_e, 2]. \end{cases} \tag{5.5.15}$$

On substitution back into (5.5.9) we obtain

$$\begin{aligned}
\mathrm{E}_{\ell,k} &= 2^{-s}(1 - \mathcal{N}_\ell^{-1}) + s\ \int_{\Delta_\ell}^{r_e} r^{-s-1}\ g_{\ell,k}(r)\ dr + s\ \int_{r_e}^{2} r^{-s-1}\ g_{\ell,k}(r)\ dr \\
&\leqslant 2^{-s}(1 - \mathcal{N}_\ell^{-1}) + s\ \mathsf{C}_3\ \int_{\Delta_\ell}^{r_e} r^{d-s-1}\ dr \\
&\quad + s\ \int_{r_e}^{2} r^{-s-1}\ \overset{*}{\mathcal{V}}_E(r)\ dr + s\ \mathsf{C}_2 \ell^{-a} \int_{r_e}^{2} r^{-s-1}\ dr. \\
&= 2^{-s}(1 - \mathcal{N}_\ell^{-1}) + \frac{s}{d-s} \mathsf{C}_3 (r_e^{d-s} - \Delta_\ell^{d-s}) \\
&\quad + s\ \int_{r_e}^{2} r^{-s-1}\ \overset{*}{\mathcal{V}}_E(r)\ dr + \mathsf{C}_2\ \ell^{-a}\ (r_e^{-s} - 2^{-s}).
\end{aligned}$$

We see that this upper bound is independent of our codepoint index k and therefore we have

$$\begin{aligned}\mathrm{E}_\ell\, \mathsf{U} \leqslant{}& 2^{-s}(1-\mathcal{N}_\ell^{-1}) + \frac{s}{d-s}\mathsf{C}_3(r_e^{d-s} - \Delta_\ell^{d-s}) \\ &+ s\int_{r_e}^{2} r^{-s-1}\ \overset{*}{\mathcal{V}}_E(r)\ dr + \mathsf{C}_2\ \ell^{-a}\ (r_e^{-s} - 2^{-s}).\end{aligned} \tag{5.5.16}$$

Using (2.11.8), we have

$$\mathcal{I}\,\mathsf{U} = \int_0^2 \mathsf{U}(r)\ D\overset{*}{\mathcal{V}}_E(r)\ dr = \mathsf{U}(2) - \int_0^2 D\mathsf{U}(r)\ \overset{*}{\mathcal{V}}_E(r)\ dr = 2^{-s} + s\int_0^2 r^{-s-1}\ \overset{*}{\mathcal{V}}_E(r)\ dr.$$

Using (5.5.13) therefore have

$$\begin{aligned}\mathrm{E}_\ell\, \mathsf{U} - \mathcal{I}\,\mathsf{U} \leqslant{}& \frac{s}{d-s}\mathsf{C}_3(r_e^{d-s} - \Delta_\ell^{d-s}) + \mathsf{C}_2\ \ell^{-a}\ r_e^{-s} \\ &- s\int_0^{r_e} r^{-s-1}\ \overset{*}{\mathcal{V}}_E(r)\ dr - 2^{-s}\big(\mathcal{N}_\ell^{-1} + \mathsf{C}_2\ \ell^{-a}\big) \\ \leqslant{}& \frac{s}{d-s}\mathsf{C}_3\ r_e^{d-s} + \mathsf{C}_2\ \ell^{-a}\ r_e^{-s} \\ ={}& \frac{s}{d-s}\mathsf{C}_3\ (\mathsf{C}_5\ \ell^{-\frac{a}{d}})^{d-s} + \mathsf{C}_2\ \ell^{-a}\ (\mathsf{C}_5\ \ell^{-\frac{a}{d}})^{-s} = \mathrm{O}\left(\ell^{\left(\frac{s}{d}-1\right)a}\right).\end{aligned}$$

$\square$

Remarks.

The technique used to prove Theorem 5.2.1 might be able to be adapted for use with any smooth compact manifold, if the potential is a function of geodesic distance in the manifold itself as opposed to Euclidean distance in the embedding space, and the normalized spherical cap discrepancy is defined using balls defined via geodesic distance.

For the proof to work properly, it would probably be necessary for the manifold to satisfy the equivalent of the standard packing argument, this time for small geodesic balls inside larger geodesic balls.

As yet, this idea is untried.

5.5.3 Riesz energy of the EQ codes

Proof of Theorem 5.4.1.

Consider $d = 1$. The regions of $\mathrm{EQ}(1, \mathcal{N})$ are $\mathcal{N}$ equal sectors of $\mathbb{S}^1$. The normalized spherical cap discrepancy $\mathrm{disc}(2, \mathcal{N})$ is $\mathcal{N}^{-1}$ rather than $2\mathcal{N}^{-1}$ because each codepoint of $\mathrm{EQP}(1, \mathcal{N})$ lies in the centre of each region. We prove this in detail in the next paragraph.

Any line in $\mathbb{R}^2$ intersects $\mathbb{S}^1$ in at most two points. Each line L of $\mathbb{R}^2$ which intersects $\mathbb{S}^1$ at two points splits $\mathbb{S}^1$ into two caps, each of which is a closed arc of $\mathbb{S}^1$. The smaller cap, which we will call S, has normalized $\mathbb{S}^1$ area (ie. normalized arc length)

$$\mathring{\sigma}(S) \leqslant \frac{1}{2}.$$

We consider the cap S and its boundary $\partial S = L \cap \mathbb{S}^1$. The boundary ∂S consists of two points which lie in either one or two regions of $\mathrm{EQ}(1, \mathcal{N})$. If both points of ∂S lie in one region then $\mathring{\sigma}(S) \leqslant \mathcal{N}^{-1}$. If the points of ∂S are in two separate regions, then call these regions R_1 and R_2 and call the corresponding codepoints $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively. For each R_i, either the codepoint $\mathbf{x}_i$ is contained in S and the area $A_i := \mathring{\sigma}(S \cap \mathsf{R}_i)$ is at least $\frac{1}{2\mathcal{N}}$ or $\mathbf{x}_i \notin S$ and $A_i < \frac{1}{2\mathcal{N}}$. In either case the normalized discrepancy of the cap S is at most $\mathcal{N}^{-1}$.

In the case of $\mathbb{S}^d$ for $d > 1$, we also consider a cap S with boundary ∂S such that $\mathring{\sigma}(S) \leqslant \frac{1}{2}$. In the cases where ∂S intersects either the North or the South pole, we can and do perturb S so that both poles are excluded, without affecting the upper bound. Thus we consider two main cases, either S contains only one pole, which we take to be the North pole, or S does not contain either pole.

1. In the case where S contains only the North pole, the equatorial image $\Pi\, S = \Pi\, \partial S$ is the whole equator $\mathbb{S}^{d-1}$. Now consider ∂S as it crosses the zones of $\mathrm{EQ}(d, \mathcal{N})$. As per the algorithm in Section 3.2, we number the zones from North to South as 0 to $\mathsf{n} + 1$. For each zone Z_i consider the equatorial image

$\Pi(Z_i \cap \partial S)$. Each non-empty image is an annulus in $\mathbb{S}^{d-1}$, except for two which are spherical caps. In any case, when we consider the normalized $\mathbb{S}^{d-1}$ area of the images we obtain

$$\sum_{i=0}^{\mathsf{n}+1} \mathring{\sigma}_{d-1}\big(\Pi(Z_i \cap \partial S)\big) = 1. \tag{5.5.17}$$

(In this proof, the notation $\mathring{\sigma}_{d-1}$ denotes the normalized area measure on $\mathbb{S}^{d-1}$.)

2. In the case where S contains neither pole, the equatorial image $\Pi\, S = \Pi\, \partial S$ has normalized area bounded by

$$\mathring{\sigma}_{d-1}(\Pi\, \partial S) \leqslant \frac{1}{2}.$$

For each zone Z_i of $\mathrm{EQ}(d, \mathcal{N})$, each equatorial image $\Pi(Z_i \cap \partial S)$ is empty, is an annulus, or may be a spherical cap. Each meridian of $\mathbb{S}^d$ which intersects S is tangential to ∂S or intersects ∂S twice. Thus when we consider the equatorial images of the intersections of ∂S with each zone Z_i we obtain

$$\sum_{i=0}^{\mathsf{n}+1} \mathring{\sigma}_{d-1}\big(\Pi(Z_i \cap \partial S)\big) \leqslant 2\ \mathring{\sigma}_{d-1}(\Pi\, \partial S) \leqslant 1. \tag{5.5.18}$$

Therefore

$$\sum_{i=0}^{\mathsf{n}+1} \mathring{\sigma}_{d-1}\big(\Pi(Z_i \cap \partial S)\big) \leqslant 1. \tag{5.5.19}$$

holds in either case.

We now analyze both cases together. Let $N(S, i)$ be the number of regions of zone Z_i which intersect ∂S. We see that $N(S, i)$ equals the number of regions of the partition $\mathrm{EQ}(d-1, \mathsf{m}_i)$ which intersect the equatorial image $Z_i \cap \partial S$, where m_i is the number of regions in zone Z_i.

Since the equatorial image $Z_i \cap \partial S$ is in general an annulus, which is the difference between two spherical caps, we can use the inductive assumption that the normalized spherical cap discrepancy of $\mathrm{EQ}(d-1,\mathsf{m}_i)$ is bounded above by

$$\operatorname{disc}(d-1,\mathsf{m}_i) \leqslant \mathsf{C}_{d-1}\mathsf{m}_i^{\frac{1}{1-d}}.$$

This yields

$$\begin{aligned} N(S,i) &\leqslant \mathsf{m}_i\Big(\overset{*}{\sigma}_{d-1}\big(\Pi(Z_i\cap\partial S)\big) + 2\operatorname{disc}(d-1,\mathsf{m}_i)\Big) \\ &\leqslant \mathsf{m}_i\ \overset{*}{\sigma}_{d-1}\big(\Pi(Z_i\cap\partial S)\big) + 2\mathsf{C}_{d-1}\mathsf{m}_i^{\frac{d-2}{d-1}}. \end{aligned} \tag{5.5.20}$$

Let $N(S)$ be the number of regions of $\mathrm{EQ}(d,\mathcal{N})$ which intersect ∂S. The bounds (5.5.19) and (5.5.20) and the estimate (3.5.74) imply that

$$\begin{aligned} N(S) &:= \sum_{i=0}^{\mathsf{n}+1} N(S,i) \leqslant \sum_{i=0}^{\mathsf{n}+1}\mathsf{m}_i\ \overset{*}{\sigma}_{d-1}\big(\Pi(Z_i\cap\partial S)\big) + 2\mathsf{C}_{d-1}\sum_{i=0}^{\mathsf{n}+1}\mathsf{m}_i^{\frac{d-2}{d-1}} \\ &\leqslant \mathsf{m}_\ominus\ \sum_{i=0}^{\mathsf{n}+1}\overset{*}{\sigma}_{d-1}\big(\Pi(Z_i\cap\partial S)\big) + 2\mathsf{C}_{d-1}(\mathsf{n}+2)\mathsf{m}_\ominus^{\frac{d-2}{d-1}} \leqslant \mathsf{m}_\ominus + 2\mathsf{C}_{d-1}(\mathsf{n}+2)\mathsf{m}_\ominus^{\frac{d-2}{d-1}} \\ &= \mathsf{C}_{\ominus,d}\ \mathcal{N}^{\frac{d-1}{d}} + 2\mathsf{C}_{d-1}(\mathsf{n}+2)\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\ \mathcal{N}^{\frac{d-2}{d}}. \end{aligned}$$

We now use the estimate (3.5.51) for $\mathcal{N} > \mathcal{N}_0(1/2)$ and (3.2.3) to obtain

$$\mathsf{n} \leqslant \pi\ \omega_d^{-\frac{1}{d}}\ \mathcal{N}^{\frac{1}{d}}$$

and therefore

$$\begin{aligned} N(S) &\leqslant \mathsf{C}_{\ominus,d}\ \mathcal{N}^{\frac{d-1}{d}} + 2\mathsf{C}_{d-1}(\pi\omega_d^{-\frac{1}{d}}\mathcal{N}^{\frac{1}{d}}+2)\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\ \mathcal{N}^{\frac{d-2}{d}} \\ &= \big(\mathsf{C}_{\ominus,d}\ + 2\mathsf{C}_{d-1}\ \pi\omega_d^{-\frac{1}{d}}\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\ \big)\mathcal{N}^{\frac{d-1}{d}} \\ &\quad + 4\mathsf{C}_{d-1}\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\ \mathcal{N}^{\frac{d-2}{d}} \\ &\leqslant \big(\mathsf{C}_{\ominus,d}\ + 2\mathsf{C}_{d-1}\ \pi\omega_d^{-\frac{1}{d}}\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\ + 4\mathsf{C}_{d-1}\mathsf{C}_{\ominus,d}^{\frac{d-2}{d-1}}\big)\mathcal{N}^{\frac{d-1}{d}}. \end{aligned}$$

For the finite number of cases where $\mathcal{N} < \mathcal{N}_0(1/2)$ we note that $N(S) \leqslant \mathcal{N}$ and so

$$N(S) \leqslant \mathcal{N} \leqslant \mathcal{N}_0(1/2)^{\frac{1}{d}}\ \mathcal{N}^{\frac{d-1}{d}}.$$

$\square$

Proof of Theorem 5.4.2.

Fix $s \in (0,2)$ and use the abbreviation $\mathsf{U} := \mathsf{U}_s$. We use the abbreviation

$$\mathrm{E}_{\mathcal{N}} := \mathrm{E}_{\mathcal{N}}\left(\mathrm{EQP}(2)\right).$$

We also fix $\mathcal{N} > 1$ and our notation drops any explicit dependence on $\mathcal{N}$ where this causes no ambiguity. For example, use $\mathbf{x}_k$ to mean $\mathbf{x}_{\mathcal{N},k} \in \mathrm{EQP}(d,\mathcal{N})$. We also assume $\mathbf{x}_k \in \mathsf{R}_k$ for k from 1 to $\mathcal{N}$.

From Definition 2.11.7 we have

$$\begin{aligned}
\mathrm{E}_{\mathcal{N}}\,\mathsf{U} &= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\frac{1}{\mathcal{N}}\sum_{j=1,j\neq i}^{\mathcal{N}}\|\mathbf{x}_i-\mathbf{x}_j\|^{-s}\\
&= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sum_{j=1,j\neq i}^{\mathcal{N}}\int_{\mathsf{R}_j}\|\mathbf{x}_i-\mathbf{x}_j\|^{-s}\,d\overset{*}{\sigma}(\mathbf{y}),
\end{aligned}$$

because

$$\int_{\mathsf{R}_j} d\overset{*}{\sigma}(\mathbf{y}) = \overset{*}{\sigma}(\mathsf{R}_j) = \frac{1}{\mathcal{N}}.$$

Since $\mathrm{diam}(\mathsf{R}_j) \leqslant \oslash_{\mathcal{N}}$, if $\mathbf{y} \in \mathsf{R}_j$ then $\|\mathbf{x}_i-\mathbf{x}_j\| \geqslant \|\mathbf{x}_i-\mathbf{y}\| - \oslash_{\mathcal{N}}$. Therefore, taking (5.4.4) into account, define

$$\mathsf{u}_{\mathcal{N}}(r) := \begin{cases} \Delta_{\mathcal{N}}^{-s} & (r \leqslant \oslash_{\mathcal{N}} + \Delta_{\mathcal{N}}),\\ (r-\oslash_{\mathcal{N}})^{-s} & (r > \oslash_{\mathcal{N}} + \Delta_{\mathcal{N}}).\end{cases} \tag{5.5.21}$$

Then

$$\begin{aligned}
\mathrm{E}_{\mathcal{N}}\,\mathsf{U} &= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sum_{j=1,j\neq i}^{\mathcal{N}}\int_{\mathsf{R}_j}\|\mathbf{x}_i-\mathbf{x}_j\|^{-s}\,d\sigma^*(\mathbf{y}) && (5.5.22)\\
&\leqslant \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sum_{j=1,j\neq i}^{\mathcal{N}}\int_{\mathsf{R}_j}\mathsf{u}_{\mathcal{N}}\left(\|\mathbf{x}_i-\mathbf{y}\|\right)d\sigma^*(\mathbf{y})\\
&= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}\int_{\mathsf{R}_j}\mathsf{u}_{\mathcal{N}}\left(\|\mathbf{x}_i-\mathbf{y}\|\right)d\sigma^*(\mathbf{y})-\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\int_{\mathsf{R}_i}\mathsf{u}_{\mathcal{N}}\left(\|\mathbf{x}_i-\mathbf{y}\|\right)d\sigma^*(\mathbf{y})\\
&= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\int_{\mathbb{S}^2}\mathsf{u}_{\mathcal{N}}\left(\|\mathbf{x}_i-\mathbf{y}\|\right)d\sigma^*(\mathbf{y})-\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\int_{\mathsf{R}_i}\Delta_{\mathcal{N}}^{-s}\,d\sigma^*_i(\mathbf{y})\\
&= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\mathcal{J}(\mathbf{x}_i;2)\ \mathsf{u}_{\mathcal{N}}-\frac{1}{\mathcal{N}}\Delta_{\mathcal{N}}^{-s}\\
&= \mathcal{J}(2)\ \mathsf{u}_{\mathcal{N}}-\mathsf{C}_{\Delta}^{-s}\ \mathcal{N}^{\frac{s}{2}-1}, && (5.5.23)
\end{aligned}$$

where in the last two steps we have used Lemma 2.11.8.

Again using Lemma 2.11.8 we have

$$\begin{aligned}
\mathcal{J}(2)\ \mathsf{u}_{\mathcal{N}} &= \frac{2\pi}{4\pi}\int_0^2 r\mathsf{u}_{\mathcal{N}}(r)\ dr\\
&= \frac{1}{2}\int_0^{\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}} r\mathsf{u}_{\mathcal{N}}(r)\ dr+\frac{1}{2}\int_{\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}}^2 r\mathsf{u}_{\mathcal{N}}(r)\ dr\\
&= \frac{1}{2}\int_0^{\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}} r\Delta_{\mathcal{N}}^{-s}\ dr+\frac{1}{2}\int_{\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}}^2 r\left(r-\oslash_{\mathcal{N}}\right)^{-s}\ dr\\
&= \frac{1}{4}\left(\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}\right)^2\Delta_{\mathcal{N}}^{-s}+\frac{1}{2}\int_{\Delta_{\mathcal{N}}+\oslash_{\mathcal{N}}}^2 r\left(r-\oslash_{\mathcal{N}}\right)^{-s}\ dr.
\end{aligned}$$

We now substitute $t := r - \oslash_{\mathcal{N}}$ to obtain

$$\begin{aligned}
\mathcal{J}(2)\ \mathsf{u}_{\mathcal{N}} &= \frac{1}{4}\left(\Delta_{\mathcal{N}} + \oslash_{\mathcal{N}}\right)^2 \Delta_{\mathcal{N}}^{-s} + \frac{1}{2}\int_{\Delta_{\mathcal{N}}}^{2-\oslash_{\mathcal{N}}} (t + \oslash_{\mathcal{N}}) t^{-s}\ dt \\
&= \frac{1}{4}\left(\Delta_{\mathcal{N}} + \oslash_{\mathcal{N}}\right)^2 \Delta_{\mathcal{N}}^{-s} + \frac{1}{2}\int_{\Delta_{\mathcal{N}}}^{2-\oslash_{\mathcal{N}}} t^{1-s}\ dt + \frac{\oslash_{\mathcal{N}}}{2}\int_{\Delta_{\mathcal{N}}}^{2-\oslash_{\mathcal{N}}} t^{-s}\ dt \\
&\leqslant \frac{1}{4}\left(\Delta_{\mathcal{N}} + \oslash_{\mathcal{N}}\right)^2 \Delta_{\mathcal{N}}^{-s} + \frac{1}{2}\int_{\Delta_{\mathcal{N}}}^{2} t^{1-s}\ dt + \frac{\oslash_{\mathcal{N}}}{2}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt \\
&= \frac{1}{4}\left(\Delta_{\mathcal{N}} + \oslash_{\mathcal{N}}\right)^2 \Delta_{\mathcal{N}}^{-s} + \frac{1}{2}\frac{1}{2-s}\left(2^{2-s} - {\Delta_{\mathcal{N}}}^{2-s}\right) + \frac{\oslash_{\mathcal{N}}}{2}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt \\
&= \frac{1}{4}\left(\Delta_{\mathcal{N}} + \oslash_{\mathcal{N}}\right)^2 \Delta_{\mathcal{N}}^{-s} + \frac{2^{1-s}}{2-s} - \frac{1}{2}\frac{{\Delta_{\mathcal{N}}}^{2-s}}{2-s} + \frac{\oslash_{\mathcal{N}}}{2}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt \\
&= \frac{1}{4}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2\ \mathsf{C}_{\Delta}^{-s}\mathcal{N}^{\frac{s}{2}-1} + \frac{2^{1-s}}{2-s} - \frac{1}{2}\frac{\mathsf{C}_{\Delta}^{2-s}}{2-s}\mathcal{N}^{\frac{s}{2}-1} + \frac{\mathsf{C}_{\oslash}}{2}\mathcal{N}^{-\frac{1}{2}}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt \\
&= \frac{2^{1-s}}{2-s} + \left(\frac{1}{4}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2\ \mathsf{C}_{\Delta}^{-s} - \frac{1}{2}\frac{\mathsf{C}_{\Delta}^{2-s}}{2-s}\right)\mathcal{N}^{\frac{s}{2}-1} + \frac{\mathsf{C}_{\oslash}}{2}\mathcal{N}^{-\frac{1}{2}}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt \\
&= \frac{2^{1-s}}{2-s} + \frac{\mathsf{C}_{\Delta}^{-s}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2 - \frac{\mathsf{C}_{\Delta}^{2}}{2-s}\right)\mathcal{N}^{\frac{s}{2}-1} + \frac{\mathsf{C}_{\oslash}}{2}\mathcal{N}^{-\frac{1}{2}}\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt. \qquad (5.5.24)
\end{aligned}$$

The integral in the last term of (5.5.24) now splits into two cases.

If $s \neq 1$ then

$$\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt = \frac{1}{1-s}\left(2^{1-s} - \Delta_{\mathcal{N}}^{1-s}\right) = \frac{1}{1-s}\left(2^{1-s} - \mathsf{C}_{\Delta}^{1-s}\mathcal{N}^{\frac{s}{2}-\frac{1}{2}}\right).$$

Therefore, from (5.5.24)

$$\begin{aligned}
\mathcal{J}(2)\ \mathsf{u}_{\mathcal{N}} &\leqslant \frac{2^{1-s}}{2-s} + \frac{\mathsf{C}_{\Delta}^{-s}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2 - \frac{\mathsf{C}_{\Delta}^{2}}{2-s}\right)\mathcal{N}^{\frac{s}{2}-1} + \frac{2^{-s}\mathsf{C}_{\oslash}}{1-s}\mathcal{N}^{-\frac{1}{2}} - \frac{\mathsf{C}_{\oslash}}{2}\frac{\mathsf{C}_{\Delta}^{1-s}}{1-s}\mathcal{N}^{\frac{s}{2}-1} \\
&= \frac{2^{1-s}}{2-s} + \frac{2^{-s}\mathsf{C}_{\oslash}}{1-s}\mathcal{N}^{-\frac{1}{2}} + \frac{\mathsf{C}_{\Delta}^{-s}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2 - \frac{\mathsf{C}_{\Delta}^{2}}{2-s} - \frac{\mathsf{C}_{\Delta}\ \mathsf{C}_{\oslash}}{1-s}\right)\mathcal{N}^{\frac{s}{2}-1},
\end{aligned}$$

and therefore, from (5.5.23)

$$\mathrm{E}_{\mathcal{N}}\,\mathsf{U} \leqslant \frac{2^{1-s}}{2-s} + \frac{2^{-s}\mathsf{C}_{\oslash}}{1-s}\mathcal{N}^{-\frac{1}{2}} + \frac{\mathsf{C}_{\Delta}^{-s}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta} + \mathsf{C}_{\oslash}\right)^2 - \frac{\mathsf{C}_{\Delta}^{2}}{2-s} - \frac{\mathsf{C}_{\Delta}\ \mathsf{C}_{\oslash}}{1-s} - 2\right)\mathcal{N}^{\frac{s}{2}-1}. \qquad (5.5.25)$$

If $s < 1$, then the second term of (5.5.25) is positive and since in this case $\mathcal{N}^{-\frac{1}{2}} > \mathcal{N}^{\frac{s}{2}-1}$, we have

$$\mathrm{E}_{\mathcal{N}} \leqslant \frac{2^{1-s}}{2-s} + \frac{2^{-s}\ \mathsf{C}_{\oslash}}{1-s}\mathcal{N}^{-\frac{1}{2}} + \mathrm{O}\left(\mathcal{N}^{\frac{s}{2}-1}\right). \tag{5.5.26}$$

If $s > 1$, then $\mathcal{N}^{\frac{s}{2}-1} > \mathcal{N}^{-\frac{1}{2}}$ and the more complicated third term of (5.5.25) dominates the second term, yielding

$$\mathrm{E}_{\mathcal{N}}\,\mathsf{U} \leqslant \frac{2^{1-s}}{2-s} + \mathrm{O}\left(\mathcal{N}^{\frac{s}{2}-1}\right). \tag{5.5.27}$$

If $s = 1$ then

$$\int_{\Delta_{\mathcal{N}}}^{2} t^{-s}\ dt = \log(2) - \log(\Delta_{\mathcal{N}}) = \log(2) - \log\left(\mathsf{C}_{\Delta}\mathcal{N}^{-\frac{1}{2}}\right) = \log\left(\frac{2}{\mathsf{C}_{\Delta}}\right) + \frac{1}{2}\log(\mathcal{N}).$$

Therefore, from (5.5.24)

$$\begin{aligned}
\mathcal{J}(2)\ \mathsf{u}_{\mathcal{N}} &\leqslant \frac{2^{1-s}}{2-s} + \frac{\mathsf{C}_{\Delta}^{-s}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta}+\mathsf{C}_{\oslash}\right)^2 - \frac{\mathsf{C}_{\Delta}^2}{2-s}\right)\mathcal{N}^{\frac{s}{2}-1} + \frac{\mathsf{C}_{\oslash}}{2}\mathcal{N}^{-\frac{1}{2}}\left(\log\left(\frac{2}{\mathsf{C}_{\Delta}}\right) + \frac{1}{2}\log(\mathcal{N})\right)\\
&= 1 + \frac{\mathsf{C}_{\Delta}^{-1}}{2}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta}+\mathsf{C}_{\oslash}\right)^2 - \mathsf{C}_{\Delta}^2\right)\mathcal{N}^{-\frac{1}{2}} + \frac{\mathsf{C}_{\oslash}}{2}\mathcal{N}^{-\frac{1}{2}}\left(\log\left(\frac{2}{\mathsf{C}_{\Delta}}\right) + \frac{1}{2}\log(\mathcal{N})\right)\\
&= 1 + \frac{\mathsf{C}_{\oslash}}{4}\mathcal{N}^{-\frac{1}{2}}\log(\mathcal{N}) + \frac{1}{2\ \mathsf{C}_{\Delta}}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta}+\mathsf{C}_{\oslash}\right)^2 - \mathsf{C}_{\Delta}^2 + \mathsf{C}_{\Delta}\ \mathsf{C}_{\oslash}\log\left(\frac{2}{\mathsf{C}_{\Delta}}\right)\right)\mathcal{N}^{-\frac{1}{2}}.
\end{aligned}$$

and therefore, from (5.5.23)

$$\begin{aligned}
\mathrm{E}_{\mathcal{N}}\,\mathsf{U} &\leqslant 1 + \frac{\mathsf{C}_{\oslash}}{4}\mathcal{N}^{-\frac{1}{2}}\log(\mathcal{N}) + \frac{1}{2\ \mathsf{C}_{\Delta}}\left(\frac{1}{2}\left(\mathsf{C}_{\Delta}+\mathsf{C}_{\oslash}\right)^2 - \mathsf{C}_{\Delta}^2 + \mathsf{C}_{\Delta}\ \mathsf{C}_{\oslash}\log\left(\frac{2}{\mathsf{C}_{\Delta}}\right) - 2\right)\mathcal{N}^{-\frac{1}{2}}\\
&\leqslant 1 + \frac{\mathsf{C}_{\oslash}}{4}\mathcal{N}^{-\frac{1}{2}}\log(\mathcal{N}) + \mathrm{O}\left(\mathcal{N}^{-\frac{1}{2}}\right).
\end{aligned} \tag{5.5.28}$$

□

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